

Lecture 24: Application of Max-flow Min-cut Theorem

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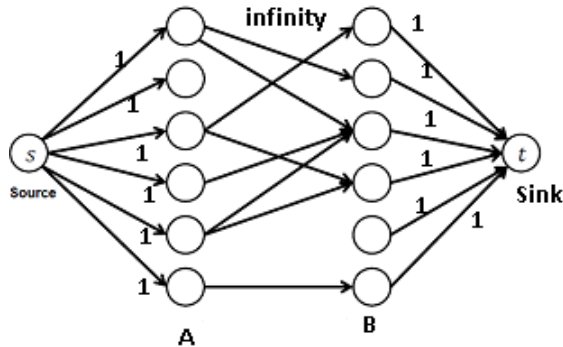
An alternative proof of Hall's Marriage theorem:

We present a proof of Hall's marriage theorem, one of the best-known results in combinatorics, using max-flow min-cut theorem.

Theorem: Suppose that G is a bipartite graph $(A; B; E)$, with $|A| = |B|$. Then G has a perfect matching iff the following condition holds: $\forall S \subseteq A, |S| \leq |N(S)|$.

Proof:

First notice that the condition above is trivially necessary for a perfect matching to exist; indeed, if we had a subset $S \subseteq A$ with $|S| > |N(S)|$, then there is no way to match up all of the $|S|$ elements of S along edges with elements in B without using some elements more than once.



Let G be a bipartite graph with bipartition (A, B) . Suppose G satisfies Hall's condition. We need to show G has a complete matching from A to B . Form a directed graph from G by adding a source vertex s , sink vertex t , joining s to each vertex in A and joining each vertex in B to t . Assign capacity 1 to each edge joining s to A and B to t . Assign capacity ∞ to each edges joining A to B . In this digraph, the maximum flow from s to t is equal to the maximum number of independent edges in G and hence is equal to the maximum size of a matching in G . We need to show this value is $|A|$. A min cut in the digraph is a set of vertices of the form $T_1 \cup T_2$ where $T_1 \subseteq A, T_2 \subseteq B$. To show that the max flow value is $|A|$, by the max flow min cut theorem it suffices to show that the min cut has value $|A|$. It's clear the min cut has size at most $|A|$ since A is a cut.

Let $S_1 = A - T_1$ and $S_2 = B - T_2$. Since $T_1 \cup T_2$ is a cut, there are no edges in G from S_1 to S_2 . Hence, all the neighbors of S_1 are in T_2 . If the min cut has size less than $|A|$, then $|T_1| + |T_2| < |A|$, and since $|T_1| + |S_1| = |A|$, we have $|T_2| < |S_1|$, which violates Hall's condition. Hence, the min cut has size equal to $|A|$. This proves that the maximum flow value is equal to $|A|$. This implies that G has $|A|$ independent edges.

Menger's Theorem:

Let s, t be a pair of distinct vertices in an undirected graph G . Then, the minimal number of edges needed to separate s from t is equal to the maximal number of edge-disjoint paths

connecting s to t .

Proof:

Let s, t be the source and sink nodes of G , respectively. Replace each of G 's undirected edges $\{x, y\}$ with pairs of edges $(x, y), (y, x)$ and let c be a capacity function on G that assigns 1 to every edge. Let f be a maximal flow on G , and S be a corresponding minimal cut. Our flow f clearly defines the maximal number of edge-disjoint paths from s to t , by identifying edges as being in a path iff f is 1 on them. Similarly, S clearly corresponds to a minimal separating set of edges; therefore, because the value of any maximum flow is the value of a minimal cut, we've completed our proof.