Discrete Mathematics 07/11/2019

Lecture 25: Linear Programming

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1 Optimization

Maximize $4x + 3y + 2z$ $\rightarrow$ $\begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

such that (i) x+y+z≤5
(ii) 2x+3y≤6
(iii) x,y,z≥0

This is called Linear Programming as the expression as well as the constraints both are linear.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

2 Standard Formulation of Linear Programming

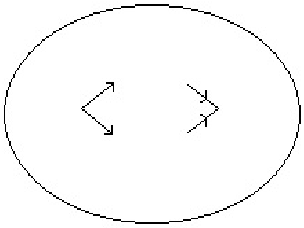
$$\begin{aligned} \text{Max} \quad & c.x \\ \text{such that } Ax &\leq b \\ x &\geq 0 \\ x &\in \mathbb{R} \end{aligned}$$

Some Algorithms to solve linear programming quickly

1. Simplex
2. Ellipsoid
3. Karmarkar

Given graph G=(V,E) c: $E \in \mathbb{R}$

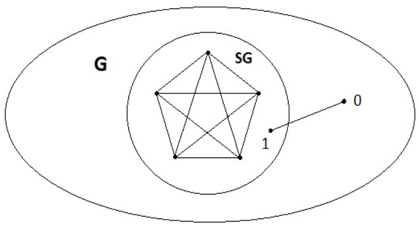
Find the max flow
$f_e \geq 0$
$f_e \in \mathbb{R}$
$\forall$ e$\in$ E
f_e is the variable that determines the amount of flow through the edge.



$$\forall v \neq s,t \sum_{u:(u,v)\in E} f_{(u,v)} = \sum_{u:(v,u)\in E} f_{(v,u)}$$

$$Max \sum_{u:(s,u)\in E} f_{(s,u)}$$

3 Finding maximum clique



$\forall u \in V\ X_u \in \{0,1\}$
edges $\forall u, v\ y_{(u,v)} \in \{0,1\}$

Max $\sum x_u$ defines maximum the vertex goes into the clique.

If $X_u = 1$ & $X_v = 1 \implies$ (u,v)∈ E

But this is not linear(LP), this is called Integer Linear Programming (ILP).

We can conclude again like,

$X_u + X_v - 1 \leq y_{u,v}$ $\quad \forall$ u,v ∈ V
$y_{u,v}$=0 $\quad \forall$ (u,v) ∉ E

Now this is LP for clique.

When variables are integers like x,y,z ∈{0,1} then it is called Integer Linear Programming(ILP) and it does not posses quick solution.

4 LP Duality

(1) Maximize (3x+4y)
x+y≤ 3 ...(i)
2x+3y ≤ 4 ...(ii)
x,y > 0
x,y ∈ ℝ Can 3x+4y be 8

Ans: On addition of equation (i) & (ii) we have 3x+4y ≤ 7
Hence 3x+4y can never be equal to 8.

(2) Maximize (3x+3y)
x+y≤ 3 ...(i)
2x+3y ≤ 4 ...(ii)
x,y > 0
x,y ∈ ℝ Can 3x+3y be 8

From question (1) we can conclude 3x+3y < 3x+4y ≤ 7 < 8

But we will like to have a tighter bound
Let us consider

α(i) + β(ii) $\alpha, \beta \geq 0$

i.e $\alpha x + 2\beta x + \alpha y + 3\beta y \leq 3\alpha + 4\beta$

such that

$(\alpha + 2\beta) \geq 3$

$(\alpha + 3\beta) \geq 3$

for   3x+3y $\leq (\alpha + 2\beta)x + (\alpha + 3\beta)y \leq 3\alpha + 4\beta$

$$i.e \quad \left.\begin{array}{ll} \text{we have to find } & \min(3\alpha + 4\beta) \\ \quad\text{such that} & (\alpha + 2\beta) \geq 3 \\ & (\alpha + 3\beta) \geq 3 \\ & \alpha, \beta \geq 0 \end{array}\right\} \text{Dual}$$

$\therefore$ Maximum value of a LP is upper-bounded by minimum value of another LP.

And interestingly here

max (3x+3y)=min$(3\alpha + 4\beta)$ [By LP Duality Theorem]

N.B.: ILP does not posses Duality.