Discrete Mathematics	5.08.2019

Lecture 4: Mathematical Induction

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1 Mathematical Induction

Mathematical induction is a proof technique used to show that a given property holds for all members of a countable inductive set. Usually we induct on $\mathbb{N}$. Assume we are trying to prove that the proposition $P(n)$ is true $\forall n \in \mathbb{N}$,a proof by induction usually involves three steps.

1. Base Case: Show that $P(0)$ is true.
2. Inductive Hypothesis: Assume P(n) is true.
3. Inductive Step: Show that $P(n) \Rightarrow P(n+1)$

Weak Induction - The above variant of induction is also known as weak induction because it uses only the assumption $P(n)$ to prove that $P(n+1)$ is true.

- If $P(n)$ is true, then $P(n+1)$ is true.

Strong Induction - In weak induction it was required that for proving $P(n+1)$, the assumption that $P(n)$ is true. But sometimes for proving $P(n)$ we need the stronger assumption that not only the immediate predecessor but all the predecessors of n are true.

- If $P(k)$ is true for all $n < k$,then $P(n)$ is true.

1.1 Induction Hypothesis and Base Cases

A important thing to note is that in the induction step one need not always show that $P(n) \Rightarrow P(n+1)$, the steps can be $P(n) \Rightarrow P(n+2)$. The examples in the previous lecture highlight this fact. But one must be careful with the base case when taking jumps in the inductive step as above.

- For example consider the inductive step $P(n) \Rightarrow P(n+2)$, this example needs two base cases,namely $P(1)$ and $P(2)$ to cover both the odd and even numbers,while inducting on $\mathbb{N}$.

The notion of induction is that one must be able to cover all the elements of the set N through the inductive step to prove that the proposition is valid for all N. The induction step can vary from problem to problem, it is arrived at based on the suitable method for the problem at hand.

1.2 Induction on Combinatorial Structures

What does induction on $P(n)$ mean?
Traditionally, induction on $P(n)$ refers only to a single proposition that is based on n. $P(n)$ doesn't have any structure that can vary with n.
But this is not the case always.Sometimes $P(n)$ refers to a class of objects not a single function or equation. In this case, care must be taken when proving the induction hypothesis from $P(n-1)$ to $P(n)$.

- For example, consider $P(n)$ referring to some property of a graph on N vertices. The graph can have any structure.

It is a common mistake while inducting on combinatorial structures to start from $P(n-1)$ and to show $P(n)$.It can lead to erroneous result as will be seen in later examples. It should be remembered that the induction step $P(n) \Rightarrow P(n+1)$ is still valid, what is changing is the method of proof. The correct way to prove is to consider a arbitary combinatorial structrue with n vertices and use the assumption of $P(n-1)$ and arrive at $P(n)$.

2 Solved Problems

2.1 Maximum number of handshakes

Problem: In a meeting of $2n$ scientists, the scientists follow a certain protocol P for shaking hands. For any three scientists A,B and C. If A and B shake hands with each other and if B and C shake hands with each other, then A and C won't shake hands with each other. Find the maximum number of handshakes that is possible in this meeting.

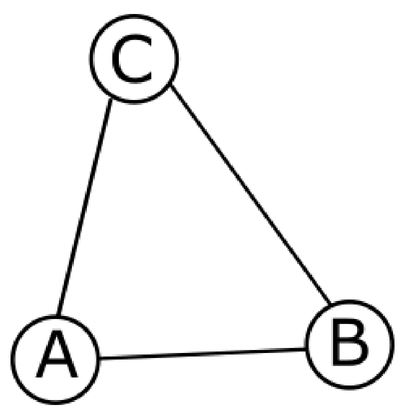

Figure 1: An combination of handshakes which violates the property.

- **Graph theoretic Viewpoint** - the above problem can be restated into a graph theory problem. The scientists can be considered as vertices and if there is a handshake between two scientists, then it can be considered as an edge. The property P is the property of the graph such that for any three vertices$A, B, C \in G$, if A and B share an edge,B and C share an edge then A and C does not share an edge. Find the maximum number of edges in such a graph.

Solution:A graph G is said to satisfy **Property P** for any three vertices $A, B, C \in G$, if there is any edge from A to B and if there is any edge from B to C, then there won't be any edge from A to C.
Consider the maximum number of edges in a graph G with $2n$ vertices that satisfy the property P is M. One direct implication of the property is that the graph should be triangle-free, it should not contain any cycles of length 3. Let us try to bound the solution within an upper and lower bound and look for some patterns.

- **Lower bound** - Any graph that has only one edge automatically satisfies the condition,therefore $M >= 1$.
- **Upper bound** - For any graph, the maximum number of edges the graph can have is equal to the number of edges in a complete graph with same number of vertices. The maximum number of edges in a graph with $2n$ vertices is $2nC2$, therefore $M <= 2nC2$.
- **Improved Lower Bound** -

  Theorem 2.1 *A graph is bipartite if and only if it has no cycles of odd length.*

  Proof. Left as an exercise for the reader □

  A graph is bipartite iff it has no odd cycles. Any graph with no odd cycles will automatically satisfy the property P of not having cycles of length 3. If we can construct a bipartite graph from the $2n$ vertices, then the maximum number of edges possible will be n^2. Therefore a bipartite graph on $2n$ vertices will give the lower bound on M. $M >= n^2$.

If we can now prove that upper bound M by n^2,then we can tell that $M = n^2$.

Theorem 2.2 (Upper Bound of M in a graph that satisfies P) *The upper bound on the number of edges in a graph with $2n$ vertices that satisfies the protocol P is n^2 i.e, $M <= n^2$*

Proof. By Induction

- **Base Case :** $P(2)$ is true. It can be easily verified that for a graph with 2 vertex the maximum number of edges 1 which is $< 1^2$.
- **Induction Hypothesis :** $P(n-1)$ is true i.e, If G is a triangle free graph on $2(n-1)$ vertices, then $E(G) <= (n-1)^2$, where $E(G)$ is the maximum number of edges in the graph.
- **Inductive Step:** Let G be a triangle free graph with $2n$ vertices.

  If there is no edge, the condition is satisfied. $P(n)$ is true.
  Otherwise, there exists atleast one edge between two vertices,consider them as u and v. Now consider the two graphs, u, v and $G' = G - u, v$. Take some vertex $w \in G'$,

- **Observation:** If u is connected to w, then w cannot not connected to v,otherwise we will produce a graph with triangle since u, v is connected.
- **Observation:**The above observation implies that all vertices $\forall w \in G'$,w is either connected to u or v but not both.

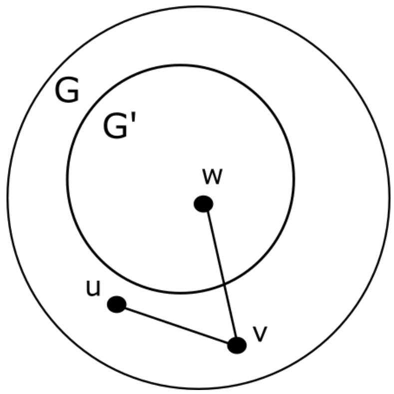

Figure 2: Graph corresponding to the observations.

Let $E(G, G')$ mean the maximum number of edges from G to G',The above observations show that,

$$E(G, G') <= 2 * (n - 1)$$

Now $E(G)$ can be written as,

$$E(G) <= E(G') + E(G', \{u, v\}) + 1$$

Now $E(G') <= (n-1)^2$ from the induction hypothesis and $E(G', \{u, v\}) <= 2*(n-1)$ can be substituted into the equaiton.

$$E(G) <= (n - 1)^2 + 2 * (n - 1) + 1$$

This can be further simplified to

$$E(G) <= n^2$$

This implies, $M <= n^2$

□

From the lower bound that $M >= n^2$ and the upper bound obtained from Theorem2.2,that $M <= n^2$, it can be concluded that

$$M = n^2$$

2.2 Wrong way of doing Induction

The proof below highlights the error in trying to prove $P(n)$ starting from $P(n - 1)$ when the induction is on combinatorial structures. It helps us to prove a false statement as true.

Conjecture 2.3 *If a graph G has $>= 3$ vertices and minimum degree of G is 2,then G has a triangle i.e an odd cycle of length* 3

Proof. By Induction,

- **Base Case:** For $n = 3$, the following graph satisfies the properties

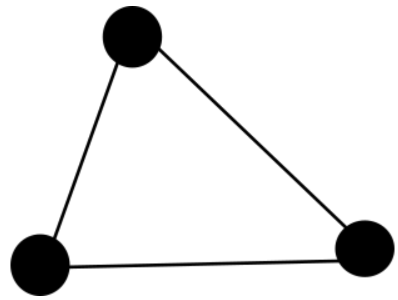

Figure 3: Base Case

- **Induction Hypothesis:** If G is a graph on $n-1$ vertices and having minimum degree of 2,then G has a triangle.

- **Induction Step:**Prove that the property is true for a graph with n vertices. Let G be a graph with $n-1$ vertices. Consider adding a new vertex a to get a graph with n vertices. Now a has to have a minimum degree of 2, so it must be connected to some two vertex m, n in G.

  - **Observation:** If m and n are connected,then these three edges form a cycle of length 3 and the condition is satisfied
  - **Observation:** Otherwise,if m and n are not connected, then since G has already a triangle addition of the new vertex will not remove the triangle,therefore the condition is still satisfied.

  From the two observations we can conclude that,

$$G(n-1) \Rightarrow G(n)$$

Therefore **conjecture 2.2** is true

□

Is it really true?. We can come up with a simple counter example. Consider the following graph

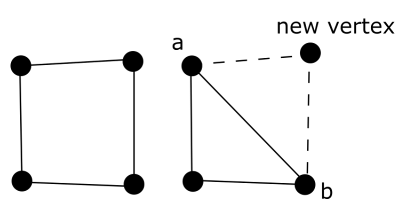

Figure 4: left :Counter example, right:Graph of 3 vertices from which a is reachable

What really happened?. The mistake was that not all graphs with n vertices and minimum degree 2 can be generated by a graph with $n-1$ vertices that satisfies a property

P. The graph in **Fig4left** with 4 vertices cannot be generated by the graph with 3 vertices that is part of the induction hypothesis(Fig3).
But the can graph in **Fig4left** can be arrived from this graph(**Fig4left**) with the edge $a - b$ removed which is not part of the inductive hypothesis as the minimum degree will not be 2.