

Lecture 6: More on Mathematical Induction

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1 Some definitions related to Graphs

Definition (Path). A path in a graph is a sequence of edges which joins a sequence of vertices.

Definition (Cycle). A cycle in a graph is path such that all the edges are distinct and the only repeated vertices are the first and last vertices.

Definition (Hamiltonian Path). A Hamiltonian path is a path in an undirected or directed graph that visits each vertex exactly once.

Definition (Hamiltonian Cycle). A Hamiltonian cycle is a Hamiltonian path that is a cycle.

Definition (Subgraph). A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Definition (Induced Subgraph). A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ and if $u, v \in V'$ then $(u, v) \in E' \Leftrightarrow (u, v) \in E$.

Definition (Clique). A complete subgraph of graph is called a clique of graph.

Definition (Independent Set). The set of vertices of graph such that there is no edge connecting any two vertices.

2 Problems

Problem 1. There is a tournament with n teams t_1, t_2, \dots, t_n . For each t_i and t_j play with each other only once and either t_i wins or t_j wins. Prove that there exists an ordering $t_{i_1}, t_{i_2}, \dots, t_{i_n}$ such that for all i in $\{1, \dots, n-1\}$, t_{i+1} defeats t_i .

Solution (Using Mathematical Induction). To solve this problem we will convert this problem into a problem in graph theory. Assume a Graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$ where each vertex v_i represents a team t_i and if v_j defeated v_i then there exists an edge (v_i, v_j) . Now the problem reduces to the following:

Every tournament has a Hamiltonian path.

We will now use Mathematical induction to prove the above statement.

Let $P(n)$: Any tournament with n vertices has a Hamiltonian path.

Base Case: Consider $P(2)$. In this case there exists two teams, and one team will win. Assume t_2 defeats t_1 . Then the Hamiltonian Path is v_1, v_2 .

Inductive Hypothesis: Let the statement $P(k)$ be true for $k \in \mathbb{N}$.

Inductive Step: Let G be a any tournament with $k+1$ vertices. Let v be a vertex of this graph. Let graph G' be formed by removing v from G . By induction hypothesis we can say that G' has a Hamiltonian path. Let this path be v_1, v_2, \dots, v_k . Using this we will prove $P(k+1)$. G when compared to G' contains vertex v in the following cases:

1. $(v, v_1) \in E(G)$: Then we can construct the Hamiltonian path in the following way v, v_1, v_2, \dots, v_k .
2. $(v_k, v) \in E(G)$: Then we can construct the Hamiltonian path in the following way v_1, v_2, \dots, v_k, v .
3. $(v, v_1) \notin E(G)$ and $(v_k, v) \notin E(G)$: We look through the edges incident to v in order (starting with edge containing v_1 , then edge containing v_2, \dots) there must come a point where the edges switch from pointing *towards* v to pointing *away* from v . So, there exists a number i , $1 \leq i \leq n-1$ such that $(v_i, v) \in E(G)$ and $(v, v_{i+1}) \in E(G)$. So the Hamiltonian path is $v_1, v_2, \dots, v_i, v, v_{i+1}, \dots, v_n$.

So, we have shown that $P(k+1)$ is true whenever $P(k)$ is true. Hence by the Mathematical Induction we can say that, Every tournament has a Hamiltonian path.

Solution (Using Median Ordering). Let $G(V, E)$ be a graph having n vertices, We consider all the possible orderings of the vertices of G . We say that an edge (v_i, v_j) is a backward edge if v_i is ahead of v_j in that ordering of vertices of G .

Let the cost of an ordering is the number of backward edges. The ordering that minimizes the number of backward edges is called median ordering i.e if u is ahead of v ordering in an ordering then u has less number of backward edges compared to v . Let the median ordering of V be:

$$v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n$$

Let v_k be any vertex(except the last) and v_{k+1} is the next vertex ahead of u_k in this ordering. Assume the directed edge (v_{k+1}, v_k) exists. If this had been the case, then v_{k+1} would have been come before v_k in the ordering, since v_{k+1} will be having more back edges, which is not true. So, only the directed edge (v_k, v_{k+1}) exists. So every median ordering is a Hamiltonian Path.

Problem 2. For natural number p and q , the Ramsey number $R(p, q)$ is defined as the smallest integer n so that among any n people, there exist p of them who know each other, or there exist q of them who don't who know each other. Prove that:

1. $R(p+1, q+1) \leq R(p, q+1) + R(p+1, q)$
2. $R(p, q) \leq \binom{p+q-2}{p-1}$

Solution. We convert this problem into a graph theoretic problem. Assume a graph $G = (V, E)$ with n vertices then we need to find that whether there exists a clique of size p or there exists a independent set of size q . (Rest of the proof is left as an exercise)