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Lecture 7: Ramsey Numbers and some basic Graph Theory

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1 Ramsey Number

Definition 1.1 *For natural numbers p and q,* ***the Ramsey number R(p, q)*** *is defined as the smallest integer n so that among any n people, there exists p of them who know each other,or there exists q of them who do not who know each other.*

We convert this problem into a graph theoretic problem. In the paradigm of graph theory, the definition is as follows:

Definition 1.2 ***R(p, q)*** *can be defined as the smallest number such that any undirected graph on R(p, q) vertices has either a clique of size p or an independent set of size q.*

We need to prove the following inequalities:

Question 1.3 ***Prove that R(p+1, q+1) $\leq$ R(p+1, q) + R(p, q+1)***

Proof. Let $R(p+1,q)+R(p,q+1)=n$.

We need to prove that for any graph G on n vertices, G either has a (p+1) size clique or a (q+1) size independent set.

Let us choose any arbitrary vertex $v \in V_G$.

Now let us define

$$N_1 = \{u \in V_G | (u,v) \in E_G\}$$

$$N_2 = \{u \in V_G | (u,v) \notin E_G\}$$

Now, either $|N_1| \geq R(p,q+1)$ or $|N_2| \geq R(p+1,q)$.

Otherwise,

$$\begin{aligned} |V_G| &= |N_1| + |N_2| + |\{v\}| \\ &\leq R(p,q+1) - 1 + R(p+1,q) - 1 + 1 \\ &< n \end{aligned}$$

which is not possible.

Case 1 - $|N_1| \geq R(p,q+1)$: This implies that there exists an independent set of size $(q+1)$ or there is a clique of size p, all vertices of which are connected to the chosen vertex v, thus forming a clique of size $(p+1)$.

Case 2 - $|N_2| \geq R(p+1,q)$: This implies that there exists a clique of size $(p+1)$ or there is an independent set of size q, which along with the chosen vertex v, forms an independent set of size $(q+1)$.

$\therefore$ G will always have either a clique of size $(p+1)$ or an independent set of size $(q+1)$. Hence Proved. □

1.1 Higher Dimensional Induction

So far we have studied induction on only one variable. Let us now consider two variables, say p and q on which we have to perform induction (Two-Dimensional Induction).

Consider Figure 1 which shows a 2-D grid for p and q.

- Our goal is to fill the entire grid.
- We need to draw diagonals and solve the recursion.
- Note that while we are at $R(p+1, q+1)$, $R(p, q+1)$ and $R(p+1, q)$ have already been evaluated.

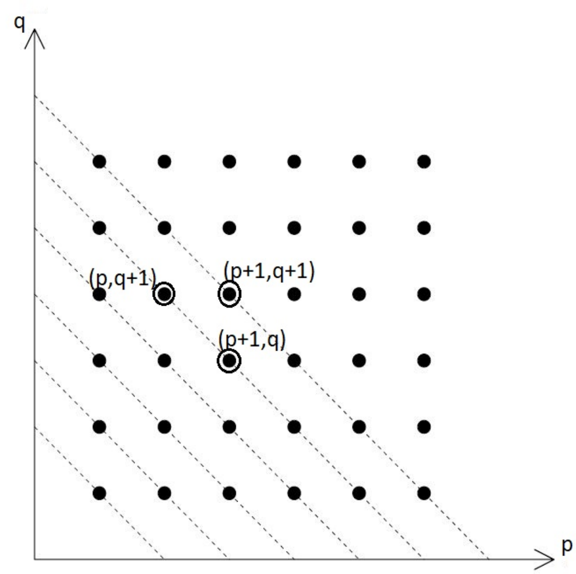

Figure 1: 2-D Grid for p and q

Question 1.4 ***Prove that R(p, q)*** $\leq \binom{p+q-2}{p-1}$

Proof. Induct on p and q.

Base Case:

$R(p, 1) = 1 = \binom{p+1-2}{p-1}$

$R(1, q) = 1 = \binom{1+q-2}{1-1}$ [$\because$ a graph with only 1 vertex is a clique by itself or an independent set by itself]

Induction Hypothesis:

Let us assume that the inequality holds for $R(x, y)$ if $x \leq p$ or $y \leq q$.

Inductive Step:

Consider $R(p+1, q+1)$.

$$\begin{aligned}\therefore R(p+1,q+1) &\leq R(p+1,q) + R(p,q+1) \\ &\leq \binom{p+1+q-2}{p+1-1} + \binom{p+q+1-2}{p-1} \\ &= \binom{p+q-1}{p} + \binom{p+q-1}{p-1} \\ &= \binom{p+q}{p} \;\; [\because \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}] \\ &= \binom{(p+1)+(q+1)-2}{(p+1)-1}\end{aligned}$$

Hence, the inequality holds for $R(p+1, q+1)$.
$\therefore$ proved by induction. □

2 Some More Problems on Graph Theory

Question 2.1 ***Prove that if G is a graph where each vertex has minimum degree 2, then it must contain a cycle***

Proof. Assume that G(V, E) does not have a cycle.
Consider the longest path in G, say,

$$u = v_0, v_1, ...v_{k-1}, v_k, v$$

Now, $\because degree(v) \geq 2, \;\; \exists \quad v_t \neq v_k$ or any of $v_0, v_1, ..., v_{k-2}, v_{k-1}$ *such that* $(v, v_t) \in E$.
Thus $v_0, v_1, ...v_{k-1}, v_k, v, v_t$ is a longer path than u.
But u is the longest path in G. So we arrive at a contradiction.
Hence G must contain a cycle. □

Question 2.2 ***Prove that if G is a graph such that every vertex has even degree, then the graph G can be written as union of edge-disjoint cycles***

Proof. Induct on the number of edges e in G.

Base Case:

Let us consider a graph having 3 edges,ie. e=3. This is because if $e < 3$, we cannot have a cycle. A connected graph in which each vertex has even degree and 3 edges must be a triangle. So it contains a cycle. Hence the statement holds for e=3.

Induction Hypothesis:

Let the statement hold $\forall\, e \leq k$.

Inductive Step:

Consider $e = k + 1$.

Now, the degree of each vertex is even and ≥ 2. Hence, there must exist a cycle, C. [Consider C to be a cycle such that no vertex apart from the starting vertex is repeated].

Consider $G - E(C)$. $G - E(C)$ must have $\leq$ k edges.

Also, $\forall v \in C$, $degree(v_{G-E(C)}) = degree(v_G) - 2$. Hence, each vertex of $G - E(C)$ still has even degree.

Now, by the Induction Hypothesis, $G - E(C)$ can be written as a union of edge-disjoint cycles.

Thus, the set of edges of G will be the disjoint union of edge sets of $G - E(C)$ and the deleted cycle, C.

Hence Proved. □

2.1 Concept of Trees

We have already seen that "If G_0 is a non-trivial graph and degree of every vertex is 0 or ≥ 2, then it must contain a cycle".

The contrapositive of the above statement is "If G_0 has no cycle then $\exists$ a vertex v_0 of degree 1".

Now consider, $G_1 = G_0 - v_0$. G_1 has no cycle and $\exists$ a vertex v_1 of degree 1.
Also, $G_2 = G_1 - v_1$. G_2 has no cycle and $\exists$ a vertex v_2 of degree 1.
In this way, we remove vertices upto $G_{n-1} = G_{n-2} - v_{n-2}$.
Now, $|G_{n-1}| = 1$.
So we have removed $n - 1$ edges and are left with an empty graph with only 1 vertex.

$\therefore$ $|E(G_0)| = n - 1$

Hence, a connected acyclic graph on n vertices has $n - 1$ edges.

Definition 2.3 *A connected, acyclic graph is called a* ***Tree***

Remark 2.4 *If we remove only an edge and a vertex from a connected graph, then it remains connected.*

Fact 2.5 *Every tree with $|V| \geq 2$ has at least 2 leaves*

Consider the longest path in a tree. The two end vertices in the path must have a degree of 1, ie. the longest path must start at a leaf and end at a leaf. Hence a tree with $|V| \geq 2$ must have at least 2 leaves.

Definition 2.6 *A disjoint collection of trees is called a* ***Forest***

Fact 2.7 *Number of edges in a Forest, $|E(Forest)| = n$ - (number of components), where n = total number of vertices*

Question 2.8 ***Given a connected graph G on ≥ 3 vertices, prove that $\exists\, u, v \in V(G)$ such that $G - u, v$ is connected***

Proof. A graph G will either have a cycle or it will be acyclic.

Case 1 - G has no cycle: Simply remove any two leaves [$\because$ by Fact 2.5, every connected, acyclic graph posesses at least 2 leaves]

Case 2 - G contains at least 1 cycle: Remove edges till G has no cycles and is still connected. Then it essentially becomes a tree.

So there must exist at least 2 leaves. Remove these, so that the graph is still connected. $\therefore$ there always exists $u,\ v\ \in\ V(G)$ *such that* $G - u, v$ is connected. $\square$