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Lecture 8: RECURSION

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1 Recurrences

In this lecture we shall study the methods of solving equations of the following form:

$$T(n) = 2T(\frac{n}{2}) + 1$$

$$T(n) = T(\frac{n}{2}) + 1$$

$$T(n) = 2T(\frac{n}{2}) + 2n$$

$$T(n) = 2T(\frac{n}{2}) + 3T(\frac{n}{3}) + 4n + \sqrt{n}$$

In the upper equations, they are not completely stated.To state a recursion relation completeltly, we have to give
(1) **Base Case**: In the upper examples, no base case i.e. T(1) is not given.
(2) A recursion relation is a function$T : \mathbb{N} \to \mathbb{N}$. In the upper examples,$\frac{n}{2}$ or $\frac{n}{3}$ are not in $\mathbb{N}$. To define in completely, we have to take floor or ceiling of them i.e. $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{3} \rfloor$.
So, the complete recursion relation will be

$$T(n) = 2T(\left\lceil \frac{n}{2} \right\rceil) + 1 \quad with \quad T(1) = 1$$

2 Solving Recurrences

We can solve recursion by

1. Guess the solution and verify by induction.
2. Guess the solution on a subset of domain, unfold the guess and verify by induction.
3. Guess and proof upper or lower bound or both.

Example 2.1 *Let*

$$T(n) = 2T(\left\lceil \frac{n}{2} \right\rceil) + 1 \quad with \quad T(1) = a$$

Now for $n = 2^k$, $\lceil \frac{n}{2} \rceil = \frac{n}{2} = 2^{(k-1)}$

$T(n) = 2T(\frac{n}{2})+1 \implies T(n) = 2(2T(\frac{n}{4})+1)+1 \implies T(n) = 4+T(\frac{n}{4}+2+1... \implies T(n) = 2^tT(\frac{n}{2^t})+2^{t-1}+2^{t-2}$

If $t = k$ *then* $T(n) = nT(1) + n - 1$, *since* $n = 2^k$.
Now from this we can guess the solution as

$$T(n) = nT(1) + n - 1$$

for all $n \in \mathbb{N}$.

Example 2.2

$$T(n) = 2T(\left\lceil \frac{n}{2} \right\rceil) + 3T(\left\lceil \frac{n}{3} \right\rceil) + 4n + \lceil \sqrt{n} \rceil$$

Suppose in the above case, we guess that the upper bound is $cn \log_2 n + d$, *where c and d are constant.*
That is
$T(n) \quad \leq \quad cn \log_2 n + d$
$\implies \; 2T(\frac{n}{2}) + 3T(\frac{n}{3}) + 4n + \sqrt{n} \quad \leq \quad cn \log_2 n + d$
Also;
$2T(\lceil \frac{n}{2} \rceil) + 3T(\lceil \frac{n}{3} \rceil) + 4n + \lceil \sqrt{n} \rceil \quad \leq \quad 2(c \lceil \frac{n}{2} \rceil \log_2 \lceil \frac{n}{2} \rceil + d) + 3(c \lceil \frac{n}{3} \rceil \log_2 \lceil \frac{n}{3} \rceil + d) + 4n + \lceil \sqrt{n} \rceil$

So, it is enough to show
$2(c \lceil \frac{n}{2} \rceil \log_2 \lceil \frac{n}{2} \rceil + d) + 3(c \lceil \frac{n}{3} \rceil \log_2 \lceil \frac{n}{3} \rceil + d) + 4n + \lceil \sqrt{n} \rceil \quad \leq \quad cn \log_2 n + d$
Now, since $2 \lceil \frac{n}{2} \rceil \leq (n+1), \log_2 \lceil \frac{n}{2} \rceil \leq (\log_2 n - 1)$ *and* $\log_2 \lceil \frac{n}{3} \rceil \leq (\log_2 n - 1)$
$2(c \lceil \frac{n}{2} \rceil \log_2 \lceil \frac{n}{2} \rceil + d) + 3(c \lceil \frac{n}{3} \rceil \log_2 \lceil \frac{n}{3} \rceil + d) + 4n + \lceil \sqrt{n} \rceil$
$\leq c(n+1)(\log_2 n - 1) + 2d + c(n+1)(\log_2 n - 1) + 3d + 4n + n$
$= 2cn \log_2 n + 2n \log_2 n - 2c(n+1) + 5d + 5n$
So, we have to show
$2cn \log_2 n + 2n \log_2 n - 2c(n+1) + 5d + 5n \leq cn \log_2 n + d$
$\implies \; cn \log_2 n + 2n \log_2 n + 4d + 5n \leq 2c(n+1)$, *which is not possible.*

Example 2.3 *Now let us suppose*

$$T(n) = 2T(\left\lceil \frac{n}{2} \right\rceil) + 3T(\left\lceil \frac{n}{3} \right\rceil) + 4n + \lceil \sqrt{n} \rceil \leq cn^2 + d;$$

where c and d are constant and T(1)=1.
Let us do this by induction.
Base Case:*For n=1, T(1)=1 gives*

$$1 \leq c + d \qquad \ldots\ldots\ldots\ldots.(1)$$

Induction Hypothes:*Suppose the given statement is true for all* $n \leq k$.
Inductive Step:*We want to prove for* $n = k$. *That is;*
$\quad 2T(\lceil \frac{k}{2} \rceil) + 3T(\lceil \frac{k}{3} \rceil) + 4k + \left\lceil \sqrt{k} \right\rceil \leq ck^2 + d$
$\implies \; 2[c(\frac{k}{2})^2 + d] + 3[c(\frac{k}{3})^2 + d] + 4k + \sqrt{k} \leq ck^2 + d \qquad$ *[byInduction Hypothes]*
$\implies \; c\frac{5}{6}k^2 + 5d + 5k \leq ck^2 + d \quad [\sqrt{k} \leq k]$
So it is enough to show
$c\frac{5}{6}k^2 + 5d + 5k \leq ck^2 + d$
$\implies \; 4d + 5k \leq \frac{c}{6}k^2$ *..........(2)*
Any pair of c and d satisfying (1) and (2) will be a valid value of c and d.

3 Asymptotic Notation

3.1 Asymptotic Similarity

Two functions f(n) and g(n) are said to be asymptotically similar if

$$\lim_{x\to\infty} \frac{f(n)}{g(n)} = 1$$

. This is notationally written as $f(n) \sim g(n)$.

3.2 $\mathcal{O}$-notation

A function T(n) is said to belong to $\Omega(f(n))$ if $\exists N_0, a, b$ such that

$$\forall n > N_0, T(n) \leq a(f(n) + b.$$

This is notationally written as $f(n) = \mathcal{O}(g(n))$.

3.3 Ω-notation

A function $T(n)$ is said to belong to $\Omega(f(n))$ if $\exists N_0, a, b$ such that $\forall\ n > N_0, T(n) \geq a(f(n) + b$. This is notationally written as $f(n) = \Theta(g(n))$.

3.4 Θ-notation

If $T(n) = \mathcal{O}(f(n))$ and $T(n) = \Omega(f(n))$; then we say $T(n) = \theta(f(n))$

3.5 o-notation

$T(n) = o(f(n))$ if

$$\lim_{x\to\infty} \frac{T(n)}{f(n)} = 0$$

3.6 ω-notation

$T(n) = \omega(f(n))$ if

$$\lim_{x\to\infty} \frac{f(n)}{T(n)} = 0$$

Example 3.1 *If $T(n) :=$ Numbers of prime $\leq n$, then by Prime Number Theorem*

$$T(n) \sim \frac{n}{\log_e} n$$

Example 3.2 ***Stirling's Approximation***

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$