

Lecture 9: Distribution of Balls into Bins

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1 Combinatorics

In this lecture we will study how many ways we can distribute n balls into k bins for different cases.

2 Possibilities

1. Balls are distinct or identical.
2. Bins are distinct or identical.
3. Ordering of the balls matters or not.
4. Empty bins allowed or not.

Based on this possibilities we consider different cases.

2.1 Note

Suppose we have 5 identical balls and 3 identical bins. Since balls are identical so how many this is the only matter of consideration. So, in this case $1+2+2$ and $2+1+2$ are the same case but $3+1+1$ is the different case.

2.2 Case 1

How many ways we can distribute n distinguishable balls into k distinct bins where ordering of the balls doesn't matter and empty bins allowed.

This is basically the number of function from a set of size n to a set of size k , i.e, k^n .

2.3 Case 2

How many ways we can distribute n distinguishable balls into k distinct bins where ordering of the balls matter and empty bins allowed.

Here we represent k bins with $k-1$ bars and we can think of n balls as n stars, so the problem boils down to stars and bars problem, i.e, how many ways we can arrange $(n+k-1)$ objects i.e, $(n+k-1)!/n!(k-1)!$. But in this case ordering matter, so the whole thing multiply with $n!$. Hence total no of ways is $(n+k-1)!/(k-1)!$.

2.4 Case 3

How many ways we can distribute n indistinguishable balls into k distinct bins where empty bins are allowed.

This problem boils down to how many ways we can choose n places out of $(n+k-1)$ places. Hence required answer is $(n+k-1)!/n!(k-1)!$.

2.5 Case 4

How many ways we can distribute n indistinguishable balls into k distinct bins where bins are non empty.

So the problem becomes the number of integer solution of the equation $x_1+x_2+\dots+x_k=n$, where $x_i \geq 1$ for all $i=1,2,\dots,k$. Where x_i is the number of ball in the i th bin.

Now $x_1+x_2+\dots+x_k -k=n-k$ this implies $y_1+\dots+y_k=n-k$, where $y_i \geq 0$ for all $i=1,\dots,k$.

So the problem boils down to choose $n-k$ places out of $(n-k+k-1)$ places. Hence required answer is $(n-1)!/(k-1)!(n-k)!$.

3 Generating Function

We use generating function only when the bins are distinct. To make our life simpler in the field of generating function we discuss binomial theorem next.

3.1 Binomial Theorem

The expansion of

$$(1+x)^n \tag{1}$$

for positive integer n is given by

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \tag{2}$$

Where

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} \tag{3}$$

3.2 Generalized Binomial Theorem

It is same as the previous Binomial theorem but it is valid $\forall n \in R$.

3.3 Example 1

How many ways we can distribute n indistinguishable balls into k distinct bins where empty bins allowed ?.

solution: The generating function of this problem $Q(x)$ be

$$Q(x) = (1+x^1+x^2+\dots)(1+x^1+x^2+\dots)\dots(k \text{ times}) = (1+x^1+x^2+\dots)^k = (1-x)^{-k} \tag{4}$$

So, the required result is coefficient of

$$x^n = (-1)^n \binom{-k}{n} = \binom{n+k-1}{n} \tag{5}$$

3.4 Example 2

How many ways we can distribute n distinguishable balls into k distinct bins where ordering of the balls matter and empty bins allowed ?.

solution: The generating function $Q(x)$ be

$$Q(x) = n!(1 + x^1 + x^2 + \dots)^k \quad (6)$$

So, the required result is coefficient of

$$x^n = (-1)^n \binom{-k}{n} n! = n! \binom{n+k-1}{n} \quad (7)$$