

Solutions of Assignment 1:

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Problem 1: Let R , S and T be three sets. Answer whether the following statements are true or false. In either case present a proof:

(a) $(R \cup S) = (R \cup T) \implies (S = T)$

Answer: FALSE

Proof. (Proof by counter example.)

Let, $R = \{1, 2, 3, 4\}$, $S = \{3\}$, $T = \{4\}$.

$$R \cup S = \{1, 2, 3, 4\}$$

$$R \cup T = \{1, 2, 3, 4\}$$

$$\therefore R \cup S = R \cup T$$

But $S \neq T$. □

(b) $(R \subseteq S) \implies ((R \cap T) \subseteq (S \cap T))$

Answer: TRUE

Proof. Let $x \in R$.

As $R \subseteq S$, $x \in S$ too.

If $x \in R \cap T$,

$x \in R$ and $x \in T$

$\implies x \in S$ and $x \in T$

$\implies x \in (S \cap T)$

$\therefore (R \cap T) \subseteq (S \cap T)$

$\therefore (R \subseteq S) \implies ((R \cap T) \subseteq (S \cap T))$. □

(c) $(S \subseteq T) \iff ((S \cap T) = S)$

Answer: TRUE

Proof. This has two parts.

Part 1: To show $(S \subseteq T) \implies ((S \cap T) = S)$

Firstly, it is obvious that always $(S \cap T) \subseteq S$.

So all we have to show is that $(S \subseteq T) \implies (S \subseteq (S \cap T))$.

Let, $x \in S$

$\therefore x \in T$ [as $S \subseteq T$]

So, $x \in S$ and $x \in T$

$\therefore x \in S \cap T$

$\therefore S \subseteq (S \cap T)$

$\therefore (S \cap T) = S$

Part 2: To show $((S \cap T) = S) \implies (S \subseteq T)$

Let, $x \in S$

$$\begin{aligned} &\implies x \in (S \cap T) \text{ [as } S \cap T = S] \\ &\implies x \in S \text{ and } x \in T \\ &\therefore x \in T \\ &\therefore S \subseteq T \end{aligned}$$

$$\therefore ((S \cap T) = S) \implies (S \subseteq T)$$

$$\text{So } (S \subseteq T) \iff ((S \cap T) = S) \quad \square$$

$$(d) (R \cup S) = (R \cup T) \iff ((S-R) = (T-R))$$

Answer: TRUE

Proof. This has two parts.

Part 1: To show $(R \cup S) = (R \cup T) \implies ((S-R) = (T-R))$.

Let, $x \in R \cup S$ and $x \notin R$.

$$\therefore x \in S - R$$

Now, $x \in (R \cup T)$

$$\therefore x \in (T - R)$$

$$\therefore (S - R) \subseteq (T - R)$$

Similarly, Let, $x \in R \cup T$ and $x \notin R$.

$$\therefore x \in T - R$$

Now, $x \in (R \cup S)$

$$\therefore x \in (S - R)$$

$$\therefore (T - R) \subseteq (S - R)$$

$$\therefore (T - R) = (S - R)$$

$$\therefore (R \cup S) = (R \cup T) \implies ((S-R) = (T-R))$$

Now we have to proof the converse.

Part 2: To show $((S-R) = (T-R)) \implies (R \cup S) = (R \cup T)$.

$(R \cup S) = R \cup (S - R)$ and similarly, $(R \cup T) = R \cup (T - R)$.

So, if $(S - R) = (T - R)$ we have $(R \cup S) = (R \cup T)$.

□

$$(e) P(S \cap T) = P(S) \cap P(T)$$

Answer: TRUE

Proof. Let, $x \in S \cap T$

$$\therefore x \in P(S \cap T)$$

Now, $x \subseteq S \cap T$
 $\implies X \subseteq S \wedge x \subseteq T$
 $\implies x \in P(S) \wedge x \in P(T)$
 $\implies x \in P(S) \cap P(T)$
 $\therefore P(S \cap T) \subseteq P(S) \cap P(T)$

Now, let assume,
 $x \in P(S) \cap P(T)$
 $\therefore x \in P(S) \wedge x \in P(T)$
 $\implies x \subseteq S \wedge x \subseteq T$
 $\implies x \subseteq S \cap T$
 $\implies x \in P(S \cap T)$

$\therefore P(S) \cap P(T) \subseteq P(S \cap T)$

$\therefore P(S \cap T) = P(S) \cap P(T)$. □

(f) $(S \times T)^c = S^c \times T^c$

Answer: FALSE

Proof. Proof by counter example.
Let, $S = \{1, 2\}$, $T = \{3, 4\}$, $U = \{1, 2, 3, 4, 5\}$
 $S \times T = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
 $|(S \times T)^c| = 21$ [as total 25 pairs are possible]
 $\bar{S} = \{3, 4, 5\}$, $\bar{T} = \{1, 2, 5\}$
 $|\bar{S}| \times |\bar{T}| = 3 \times 3 = 9$
 $9 \neq 21$. □

Problem 2: Prove or Disprove:

(a) The Cartesian product is associative.

Answer: FALSE

Proof. Proof by counter example. Let, $A = \{1\}$
 $((A \times A) \times A) = ((1, 1), 1)$.
 $(A \times (A \times A)) = (1, (1, 1))$.
But they are not equal. □

(b) The Cartesian product is commutative.

Answer: FALSE

Proof. Proof by counter example. Let, $A = \{1, 2\}$, $B = \{3, 4\}$
 $\therefore A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
and $B \times A = \{(3, 1), (4, 1), (3, 2), (4, 2)\}$
 $\therefore A \times B \neq B \times A$. □

- (c) The relation “is connected to” for a pair of vertices in an undirected graph is an equivalence relation.

Answer: TRUE

Proof.

- Reflexive: Let V is the set of vertices, $v \in V$.
Now, v is connected to v .
 $\therefore vRv$ exists.
- Symmetric: Let $u, v \in V$.
Now, if uRv (u is connected to v) then vRu (v is connected to u) by going through the edges in the reverse order.
- Transitive: Let $u, v, w \in V$.
Now, if uRv and vRw , then uRw , by first going from u to v and then going from v to w ,

\therefore It is an equivalence relation.

□

- (d) The set \mathbb{Q}^+ is countable.

Answer: TRUE

Proof. To show that \mathbb{Q}^+ is countable we need to show a 1-1 mapping from \mathbb{Q}^+ to \mathbb{N} .

If $\frac{p}{q} \in \mathbb{Q}^+$, where $p, q \in \mathbb{N}$ and $q \neq 0$ then let us define the bijection as follows.

$$F\left(\frac{p}{q}\right) = 2^p(2q - 1)$$

All we need to prove is that the function F is 1-1.

Let us prove by contradiction. Let $F(p/q) = F(r/s)$.

Then $2^p(2q - 1) = 2^r(2s - 1)$.

Since $q, s \geq 1$ so $(2q - 1)$ and $(2s - 1)$ are both non-zero.

The maximum power of 2 that divides LHS is 2^p while the maximum power of 2 that divided RHS is 2^r .

So if $2^p(2q - 1) = 2^r(2s - 1)$ then p must be equal to r .

And in that case we have $(2q - 1) = (2s - 1)$ which means $q = s$.

So if $F(p/q) = F(r/s)$ then $p = r$ and $q = s$.

So F is a 1-1 function and hence \mathbb{Q}^+ is countable.

□

Problem 3: Is the relation “is at least as popular as” a valid ordering and if so is it a partial ordering or a total ordering?

Proof.

Proof of non-transitive: (by counter example)

Let us assume it is transitive.

Therefore if there are 3 contestants A, B and C. Then if $A > B$ and $B > C$ then it

must be $A > C$.

Let there are 3 voters, namely 1, 2 and 3. Their preferences for contestants are as follows in Table.

1	2	3
a	b	c
b	c	a
c	a	b

Table 1: Preference

Now from column 1 and 3, $a > b$, from column 1 and 2, $b > c$.

But from column 2 and 3, $c > a$.

\therefore it is non transitive. So there is no valid ordering. \square

Problem 4: **Let R be a relation on S . Prove that $\sum_{x \in S} |N^+(x)| = \sum_{x \in S} |N^-(x)| = m$.**

Proof. Consider the relation as a directed graph.

\therefore if $(x,y) \in R$ then there is an edge $x \rightarrow y$.

Suppose the out-degree of x is k . Then these k edges incident on some other nodes.

\therefore out-degree of x adds the same to the total in-degree

\therefore if outdegree of x is k then for entire graph, out degree increases k and indegree increases k too.

$\therefore \sum_{degree} x = indegree = outdegree$

Now $|R| = m$, therefore total m edges. Each edge is connected with some node.

Consider vertex x_1 , let the edge is m_1 , for x_2 , let it be m_2 , and so on ...

$\therefore m_1 + m_2 + \dots = m$ But $m_1 =$ Out-degree of x_1 , $m_2 =$ Out-degree of x_2 , and so on.

\therefore sum of total out-degree $= m$. Similarly for in-degree.

So, $\sum_{x \in S} |N^+(x)| = \sum_{x \in S} |N^-(x)| = m$. \square

Problem 5: **How many functions are there from a domain of size n to a range of size m ?**

For each element of domain ($=n$) we can have m different functions.

$D = \{x_1, x_2, \dots, x_n\}$

$R = \{y_1, y_2, \dots, y_m\}$

The set of functions $S = \{f(x_1), f(x_2), \dots, f(x_n)\}$

Now, $f(x_1)$ can be chosen in m ways, $f(x_2)$ can be chosen in m ways, and so on..

The number of functions =

$m * m * m * \dots * m$ [n times] $= m^n$

Problem 6: **How many one to one functions are there from a domain of size n to a range of size m ?**

There can be two cases.

Case 1: If $m \geq n$:

There will be m ways to map 1st element, $m - 1$ ways to map 2nd element, $m - 2$ ways to map 3rd element, ... , $(m - n + 1)$ way to map n th element.

So, $m*(m-1)*(m-2)*\dots*(m-n+1) = m!/(m-n)! = {}^m P_n$ number of into functions.

Case 2: If $m < n$:

Not possible.

Problem 7: How many onto functions are there from a domain of size n to a range of size m ?

Let S be the domain of size n and R be the range of size m , with $m \leq n$. If $m > n$ then the number of onto functions from S to R is 0.

Let $R = \{y_1, \dots, y_m\}$.

Any function f from S to R is defined by the sets $f^{-1}(y_1), \dots, f^{-1}(y_m)$.

If f is onto we have the additional property that for all i $f^{-1}(y_i)$ is non-empty.

So the number of onto functions from S to R is the number of ways the elements of S can be partitioned into m non-empty subsets.

This number is

$$\sum_{i=0}^m (-1)^i \binom{m}{i} (m-i)^n.$$

Problem 8: If f is a function from the set $D = \{1, 2, 3, \dots, 100\}$ to $R = \{1, 2, \dots, 200\}$. How many decreasing functions are there from D to R and how many non-increasing functions are there from D to R .

If f is an decreasing function then the function is defined by the set $\{f(1), \dots, f(100)\}$ and given a subset of R of size n there is an unique decreasing function. So the number of decreasing function is $\binom{200}{100}$.

Any function f from D to R is defined by the sets $f^{-1}(1), \dots, f^{-1}(200)$.

If f is a non-increasing function we have the additional property that for all $i, j \in R$ if $i < j$ then all the elements in $f^{-1}(j)$ is before all the element in $f^{-1}(i)$.

So such a non-increasing function is just defined by the size of $f^{-1}(1), \dots, f^{-1}(200)$.

So the number of onto functions from S to R is the number of ways 100 identical objects can be partitioned into 200 subsets. The number is $\binom{299}{100}$.