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| **Discrete Mathematics** | DATE : 11 Sep 2019 |

# Mid Semester Solutions : Q1 to Q3

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**1. From the definition of the asymptotic notations prove that if $T(n) = aT(n/b)+O(n^c)$ , where c $< log_b a$ then**

$$T(n)=\Theta(n^{log_b a})$$

## Solution:

*Reference : Section 4.4 Introduction to algorithm : Cormen*
*Under the assumption that n is an exact power of b > 1*

Let $a \geq 1$ $b > 1$ and $f(n) = O(n^c)$ & $T(1) = \Theta(1)$

Iterating the recurrence,
$T(n) = f(n) + aT(n/b)$
$= f(n) + af(n/b) + a^2T(n/b^2)$
$= f(n) + af(n/b) + a^2f(n/b^2) + ... + a^{log_b n-1}f(n/b^{log_b n-1})+a^{log_b n}T(1)$

Since $a^{log_b n} = n^{log_b a}$, the last term of the expression

$$a^{log_b n}T(1) = \Theta(n^{log_b a})$$

Using the boundary condition $T(1) = \Theta(1)$, the remaining terms can be expressed as the sum

$g(n) = \sum_{j=0}^{log_b n-1} a^j f(n/b^j)$ ...(i)

Thus, $T(n) = \Theta(n^{log_b a}) + \sum_{j=0}^{log_b n-1} a^j f(n/b^j)$ ...(ii)

Now, we have $f(n) = O(n^c) \implies f(n/b^j) = O((n/b^j)^c)$ ...(iii)
Substituting (iii) in (i) yields

$$g(n) = O\left( \sum_{j=0}^{log_b n-1} a^j (n/b^j)^c \right)$$

We bound the summation within the O notation by factoring out terms and simplifying, which leaves an increasing geometric series

$\sum_{j=0}^{log_b n-1} a^j (n/b^j)^c = n^c \sum_{j=0}^{log_b n-1} (\frac{ab^{log_b a-c}}{b^{log_b a}})^j$

$= n^c \sum_{j=0}^{log_b n-1} (b^{log_b a-c})^j$

$= n^c \sum_{j=0}^{log_b n-1} (\frac{b^{(log_b a-c)(log_b n)}-1}{b^{log_b a-c}-1})$

Since b & ($log_b$a - c) are constant, the last expression reduces to $n^c$ O($n^{log_b a-c}$) = O($n^{log_b a}$)
which yields, g(n) = O($n^{log_b a}$)
Now, we use the bounds to evalute equation (ii),

$$\begin{aligned} T(n) &= \Theta(n^{log_b a}) + O(n^{log_b a}) \\ &= \Theta(n^{log_b a}) \end{aligned}$$

**Hence proved.**

To prove above for any arbitrary integer $n$ and not just restricted to be exact power of $b$ extend our analysis to the situation in which floor and ceiling are used in the recurrence.
Obtaining a lower bound

$$T(n) = aT(\lceil n/b \rceil) + f(n)$$

and upper bound on

$$T(n) = aT(\lfloor n/b \rfloor) + f(n)$$

and lower bounding the recurrence is similar to upper bounding the recurrence, so we shall only present in this later bound
Iterating the recurrence we obtain,

$$T(n) = \Theta(n^{log_b a}) + \sum_{j=0}^{\lfloor log_b n \rfloor -1} a^j f(n^j)$$

which is much the same as equation(ii), Now evaluate the summation

$$g(n) = \sum_{j=0}^{\lfloor log_b n \rfloor -1} a^j f(n_j)$$

in manner analogous to above proof and we have $f(n) = \Theta(n^{log_b a})$. If we can show that $f(n_j) = O(\frac{n}{a^j})^c$ then we are done.

$$f(n_j) \leq d(\frac{n}{b_j} + \frac{b}{b-1})^c \quad \text{(from finding j element in the sequence)}$$

$$\leq O(\frac{n}{a^j})^c$$

**2. Prove or Disprove the following set of asymptotic relations :**

**(a)** $(\mathbf{2.9})^{log_2 n} = \Theta(\mathbf{n}^{log_2 3})$

**(b)** $\mathbf{log\ log\ n} = \Omega((\mathbf{log\ log\ log\ n})^{logloglogn})$

**(c)** $\mathbf{n}^4 \sim (\mathbf{1\text{-}1/n})^n \mathbf{n}^3$

**(d)** $\mathbf{n}^{10(loglogn)^{100}} = \Theta((\mathbf{log\ n})!)$

**(e)** $\mathbf{log}\binom{2n}{n} = \mathbf{o}(\mathbf{n}^n)$

## Solution:

(a) We have $(2.9)^{log_2 n} = \text{n}^{log_2 (2.9)}$. Since $\log_2(2.9) \leq \log_2 3$, so $\frac{(2.9)^{log_2 n}}{n^{log_2 3}} = \text{n}^{log_2(2.9) - log_2 3}$ which tend to 0 as n goes to infinity.
So $(2.9)^{log_2 n} = o(\text{n}^{log_2 3})$ and hence the given statement is not true.

(b) Let's take x= log log log n.

Now, $$e^x = o(x^x)\ \forall \text{ increasing function } x.$$
$$\therefore \text{log log n} = O((\text{log log log n})^{logloglogn})$$

Hence disproving the given statement.

(c) Let x = $n^4$ and y = $(1 - 1/\text{n})^n n^3$

Then log x = 4 log n and log y = n log(1 -1/n) + 3 log n

$\lim_{n\to\infty} \frac{logy}{logx} = \lim_{n\to\infty} \frac{nlog(1-1/n)}{4logn} + \frac{3logn}{4logn} = \frac{3}{4}$

Hence, they are not asymptotically equal.

(d) For the given statement to be true , $\exists$ c $\ni n^{10(loglogn)_{100}} \leq$ c(log n)! for all n<$N_0$ for some $N_0 \in \mathbb{N}$.
Also since (log n)! <$(\text{log n})^{logn}$ so for the given statement to be true, $\exists$ c $\ni n^{10(loglogn)^{100}}$ $\leq$ c$(\text{log n})^{logn}$ for all n >$N_0$ for some $N_0 \in \mathbb{N}$

$$\text{But } \lim_{n\to\infty} \frac{n^{10(loglogn)^{100}}}{(logn)^{logn}} = \infty$$

So, no such c exits, hence the given statement is false.

(e) $\binom{2n}{n} = \frac{(2n)!}{n!n!} \sim \frac{4^n}{\sqrt{\pi n}}$ [Using Striling's Approximation]

Now, $\lim_{n\to\infty} \frac{log(\frac{4^n}{\sqrt{\pi n}})}{n^n} = \lim_{n\to\infty} \frac{nlog4 - \frac{1}{2}log(\pi n)}{n^n}$

$= \lim_{n\to\infty} \frac{log4 - \frac{1}{2\pi n}}{n^n(logn+1)}$ [Using LHópital's Rule ]

$= 0$

Hence the above statement is true.

**3. The *Lucas Sequence* 1, 3, 4, 7, 11, 18, 29, ... is defined by $\mathbf{a}_1 = \mathbf{1}$, $\mathbf{a}_2 = \mathbf{3}$, $\mathbf{a}_n = \mathbf{a}_{n-1} + \mathbf{a}_{n-2}$. Prove that $\mathbf{a}_n = \mathbf{O}(\mathbf{1.75}^n)$.**

## Solution:

Given $a_n = a_{n-1} + a_{n-2}$ , $\forall n \geq 3$ $a_1 = 2$, $a_2 = 3$.
Using generating functions to solve it,
Define, $A(x) = \sum_{i=1}^{\infty} a_i x^i$ ...(i)

$= x + 3x^2 + \sum_{i=3}^{\infty} a_i x^i$

$= x + 3x^2 + \sum_{i=1}^{\infty} ( a_{i-1} + a_{i-2} ) x^i$

$= x + 2x^2 + x\sum_{i=1}^{\infty} a_i x^i + x^2\sum_{i=1}^{\infty} a_i x^i$

$= x + 2x^2 + xA(x) + x^2 A(x)$

Therefore, $A(x) = -\frac{x+2x^2}{1-x-x^2}$ ...(ii)

The roots of the denominator are $\alpha = -\frac{1+\sqrt{5}}{2}$ and $\beta = \frac{-1+\sqrt{5}}{2}$

We can write $A(x)$ as $A(x) = -x(\frac{\gamma}{x-\alpha} + \frac{\delta}{x-\beta})$ ... (iii)

From (ii) & (iii) we get $\gamma = 1$ , $\delta = 1$,
$\therefore A(x)$ as $A(x) = -x(\frac{1}{x-\alpha} + \frac{2}{x-\beta})$

$= -x\{\frac{1}{\alpha(-\beta x-1)} + \frac{1}{\beta(-\alpha x-1)}\}$ $[\because \alpha\beta = 1]$

$= x\{\frac{1}{\alpha(\beta x+1)} + \frac{1}{\beta(\alpha x+1)}\}$

$= x\{\frac{1-x\beta+x^2\beta^2-x^3\beta^3+...}{\alpha} + \frac{1-x\alpha+x^2\alpha^2-x^3\alpha^3+...}{\beta}\}$ ...(iv)

usiing binomial expansion of $(1 + x)^{-1}$

Equating (i) and (iv), we get $a_n = \frac{(-\beta)^n}{\alpha} + \frac{(-\alpha)^n}{\beta}$

$$= \frac{(1-\sqrt{5})^n + (1+\sqrt{5})^n}{2^n}$$

$$< \frac{(-1.236)^n + (3.236)}{2^n}$$

$$< (0.168)^n$$

$$< (1.618)^n$$

$$< 2(1.618)^n$$

$$< 2(1.75)^n$$

Hence , $a_n = O((1.75)^n)$

**Alternatively**,

Given $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$, we shall show, $a_n \leq$ c$(1.75)^n$

We proceed by induction on n,

Base Case : $a_1 \leq$ c.(1.75) , $a_2 \leq$ c.$(1.75)^2$ valid for c $\geq$ 1

Inductive hypothesis: We assume, $a_i \leq$ c$(1.75)^i$ For $i \leq$ n-1

$$a_n = a_{n-1} + a_{n-2}$$

$$\leq c(1.75)^{n-1} + c(1.75)^{n-2}$$

$$= c(1.75)^{n-2}(2.75)$$

$$< c(1.75)^n \quad \text{for } c > 1.$$

Hence , $a_n = O((1.75)^n)$