Discrete Mathematics DATE : 7/10/19

Mid Semester Solutions : Q4 to Q7

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4. prove or disprove
a) Let x,y $\in$ R if $y^3 + yx^2 \leq x^3 + xy^2$ then y$\leq$x
b)If for a positive integer n, 2^n-1 is a prime then n is a prime
c) if a and b are two number that are not rational then a^b is also not rational. (Hint: Consider $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$).

Solution:

a) $y^3 + yx^2 \leq x^3 + xy^2$
$\Rightarrow$ y(y^2 + x^2) - x(x^2 + y^2) $\leq$ 0
$\Rightarrow$ (y^2 + x^2) ($y - x$) $\leq$ 0
$\Rightarrow$ As (y^2 + x^2)$\geq$0 so, y-x$\leq$0
$\Rightarrow$ y$\leq$x

b) To show: If 2^n-1 is prime from some positive integer n, then n is a prime.

Since the hypothesis requires $n \geq 2$ to be true, one may assume that. we proove the contrapositive of this statement i.e.
if n$\geq$2 is not prime then 2^n-1 is not prime either. Suppose n$\geq$2 is not prime. Then there exist integers x,y>1 such that n=xy.
One has $2^n - 1 = 2^{yx} - 1 = (2^y)^x - 1 = (2^y - 1)(2^{y(x-1)} + 2^{y(x-2)} + \cdots + 2^{y.1} + 2^{y.0})$. Since $y > 1$ the first factor $2^y - 1$ is > 1, and since $x > 1$ that factor is less than $2^n - 1$.

Since $2^n - 1$ has a proper divisor $2^y - 1$ greater than 1, we have shown that $2^n - 1$ is composite, establishing the contrapositive.

Remark: The contrapositive statement proved remains true if powers of 2 are replaced throughout by powers of some integer $a \geq 2$. However with that change it is no longer true that the original hypothesis requires $n \geq 2$, as $n = 1$ might work. Therefore such a generalisation of the original statement requires an explicit additional hypothesis $n \geq 2$.
Reference: https://math.stackexchange.com/

5. Prove that at a party with at least two people, that there are two people who know the same number of people there (not necessarily the same people - just the same number) given that every person at the party knows at least one person. (Note that nobody can be his or her own friend.)

Solution:

Let us assume that in a party with n people, each people knowing at least one person, there are no such pair of persons who know the same number of people.Suppose person p_i knows i number of people. Since the value of i varies with each different person, we say that each number of people know unique number of other people p_1 knows one person , p_2 knows two person ..... p_n knows n persons. However this is not possible, since the maximum number of persons a person can know is n-1.
Therefore P_n must know a number of person less than n , say equal to k ; k <n But pk also knows k number of people. Hence, $\Longrightarrow \Longleftarrow$
Therefore at least two people know equal number of people.
Please note that this can be directly attained by using the Pigeon Hole Principle.

6. A tournament is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph. That is, it is an orientation of a complete graph, or equivalently a directed graph in which every pair of distinct vertices is connected by a single directed edge.
(a) For any given n, give an example of a tournament which has no directed cycle.
(b) Prove that a tournament has a directed 3-cycle if and only if it has a directed cycle.

Solution:

Convention used in the solution : In a directed graph, if a and b are vertices, (a, b) implies there is a directed edge from a to b,pointing from a towards b.

a)Consider the tournament of three vertices a,b,c with the edges $(a, b), (b, c), (a, c)$. This is a valid tournament without a directed cycle.

b) Proof by induction for part 1:
If there is a directed cycle, it implies that a 3 cycle is present

Induction on the number vertices in the cycle , or the length of the cycle n.
Base case : For n=3 , we trivially satisfy the condition.
Let this hypothesis be true for n=k [Weak Induction].
Now consider a tournament of k+1 vertices that form a cycle. For a specific vertex v_i , we have an edge (v_{i-1}, v_i) and an edge (v_i, v_{i+1}). There is also an edge from v_{i-1} to v_{i+1} either in the form of (v_{i+1}, v_{i1}) or (v_{i-1}, v_{i+1}). We encounter two cases here:
i)The edge is (v_{i+1}, v_{i1}). We have a three cycle in this case:$(v_{i-1}, v_i), (v_i, v_{i+1}), (v_{i+1}, v_{i-1})$.
ii)The edge is (v_{i-1}, v_{i+1}). We remove v_i and all edges incident on it. Note that this means that we have a directed cycle of length k now, which ,by our assumption contains a 3 cycle. Clearly this 3 cycle has no involvement of v_i , so if we add back v_i now, this 3 cycle is still intact. Hence our assumption holds for $n = k + 1$.
Thus proved.

Proof for part 2: If there is a 3 cycle, the tournament has a directed cycle
Trivially proved.

7. How many ways can you distribute n identical balls into 2k distinct bins such that exactly k of the bins get odd number of balls and remaining k bins get even number of balls.

Solution:

For k bins having even number of balls $(x^2 + x^4 + x^6....)^k$
For k bins having odd number of balls $(x^1 + x^3 + x^5....)^k$
For any k bins to have even number of balls we have to chose those k bins out of 2k bins i.e. $\binom{2k}{k}$
Coefficient of x^n in the expansion of $[\binom{2k}{k} \times (x^0 + x^2 + x^4 + x^6....)^k \times (x^1 + x^3 + x^5....)^k]$

$\Rightarrow$ $[\binom{2k}{k} \times (\frac{1}{1-x^2})^k \times (\frac{x}{1-x^2})^k]$

$\Rightarrow$ $[\binom{2k}{k} \times x^k \times (\frac{1}{1-x^2})^{2k}]$

$\Rightarrow$ $[\binom{2k}{k} \times x^k \times (1 - x^2)^{-2k}]$

$\Rightarrow$ $[\binom{2k}{k} \times x^k \times \sum_r \binom{-2k}{r} x^{-4k-2r}]$

$\Rightarrow$ we need to calculate coefficient of x^{n-k} in$[\binom{2k}{k} \times \sum_r \binom{-2k}{r} x^{-4k-2r}]$.