Discrete Mathematics 30/09/2019

Mid Semester exam Solution Q8-Q11

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Q.8 Consider the functions from $\{1, 2, ..., n\}$ to $\{1, 2, ..., k\}$.

(a) How many functions are there from $\{1, 2, ..., n\}$ to $\{1, 2, ..., k\}$.
Sol.
Let A=$\{a_1, a_2, ..., a_n\}$ (in here it is $\{1, 2, ..., n\}$)
B= $\{b_1, b_2, ..., b_k\}$ (in here it is $\{1, 2, ..., k\}$)
a function f assigns each element a_i of A to an element $b_i = f(a_i)$ of B. There are k possible ways for each element of A(as no restrictions here).Hence by rule of product ,we have

$$k * k * k * k..... * k = k^n$$

this are the possible number of functions.

(b) How many increasing functions are there from $\{1, 2, ..., n\}$ to $\{1, 2, ..., k\}$.
Sol.
Let A=$\{a_1, a_2, ..., a_n\}$ (in here it is $\{1, 2, ..., n\}$)
B= $\{b_1, b_2, ..., b_k\}$ (in here it is $\{1, 2, ..., k\}$)
Here increasing function(strictly) means that for function f we have
for any a_i, a_j in A if $a_i < a_j$ then $f(a_i) < f(a_j)$.
So,it is clear that B must have at least n number of elements .So that it would map each of the elements from A(i.e $n \leq k$).
If f is decreasing function then the function defined by the set $\{f(1), f(2), ..., f(n)\}$ and this is sunset of B of size n from k which must have a unique increasing order for each n selected.
So,number of increasing functions $\binom{k}{n}$.

(c) How many non-decreasing functions are there from $\{1, 2, ..., n\}$ to $\{1, 2, ..., k\}$.
Sol.
Let A=$\{a_1, a_2, ..., a_n\}$ (in here it is $\{1, 2, ..., n\}$)
B= $\{b_1, b_2, ..., b_k\}$ (in here it is $\{1, 2, ..., k\}$)
Here non-decreasing function means that for function f we have
for any a_i, a_j in A if $a_i < a_j$ then $f(a_i) \leq f(a_j)$.
Now,here if we have a non-decreasing function f then there could be possibility $f(a_i) = b_j$,$f(a_{i+1}) = b_j$,$f(a_{i+2}) = b_j$
where a_i,a_{i+1},a_{i+2} are consecutive.This just means that we can partition A with n elements into k parts,each partition corresponds to a group of values in the domain mapping to a single element in the range.
or in other words ,this is the same as number of ordered partition of n into exactly k parts(some parts may be 0).

so this should be

$$\binom{n+k-1}{n}$$

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Q.9 Let G be a simple undirected graph .
(a)Prove that the number of vertices with odd degree in G is even.
Sol.
Let G be a graph with e edges and n vertices.Since each edge incident on two vertices (as simple undirected graph) it contributes 2 in sum of degrees of vertices in graph G.Thus,

$$\sum_{i=1}^{n} degree(v_i) = 2e$$

In this let $i = 1$ to $i = r$ (in total r vertices) be the vertices with the even degree and remaining $i = r + 1$ to $i = n$ (which is $n - r$ vertices) with odd degree.
So,

$$\sum_{i=1}^{n} degree(v_i) = \sum_{i=1}^{r} degree(v_i) + \sum_{i=r+1}^{n} degree(v_i)$$

here,
$\sum_{i=1}^{n} degree(v_i)$=even as it is twice the edge.
$\sum_{i=1}^{r} degree(v_i)$=even as each vertex in here has even degree.
So,

$$\sum_{i=r+1}^{n} degree(v_i) = \sum_{i=1}^{n} degree(v_i) - \sum_{i=1}^{r} degree(v_i)$$

which means $\sum_{i=r+1}^{n} degree(v_i)$ should be even but here each of the vertices degree is odd.So the number of terms must be even.In other words vertices of odd degree must be even .
Hence Proved.

(b)Prove that if u is a vertex of odd degree in a graph,then there exists a path from u to another vertex v of the graph where v also has odd degree.
Sol.
For a graph G, Pick an arbitrary vertex u of odd degree.Take the subset of the graph that is connected to u via some path i.e connected component of G that contains u.Call it G_u. Since G_u is a graph itself, it must have total degree even.(as proved in this question part (a)) which means that there must be another vertex v of odd degree in G_u.Since G_u is connected there must exist a path between u and another vertex v.
This could be done to any other odd degree vertex in G.
Hence Proved.

Q.10 Let $p^*(k)$ be the number of ways to partition the integer k into distinct integers.and $p_o(k)$ be the number of ways to partition integer k into odd integers.Prove that $p^*(k) = p_o(k)$.
Sol.

Let $A(k)$ be generating function that stands for the number of ways to partition into distinct integers.
Example for this when $n = 6$ we can get $\{6, 5+1, 4+2, 3+2+1\}$. Clearly,

$$A(x) = (1+x)(1+x^2)1+x^3)....$$

$$A(x) = \prod_{i=1}^{\infty}(1+x^i)$$

This is basically to count partitions with distinct parts.For each term $(1+x^i)$ we should choose in i that weather to use the particular i once or not at all.

Let $B(x)$ be generating function that stands for the number of ways to partition integer k into odd integers.
Example for this when $n = 6$ we can get $\{5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1\}$.
So,

$$B(x) = (1+x+x^2+x^3....)(1+x^3+x^6+x^9..)...$$

$$B(x) = \prod_{i=1}^{\infty}\frac{1}{(1-x^{2i-1})}$$

Here we use the factors for odd values of i to get the generating function.Like number of times 1 could come is represented by $(1+x+x^2+x^3....)$ similarly for $3, 5...$

If we try to rewrite $A(x)$ it is

$$= (1+x)(1+x^2)(1+x^3)....$$

if we multiply and divide each term of $(1+x^i)$ with $\left\{\frac{1-x^i}{1-x^i}\right\}$ then

$$= \left\{\frac{1-x^2}{1-x}\right\}\left\{\frac{1-x^4}{1-x^2}\right\}\left\{\frac{1-x^6}{1-x^3}\right\}.....$$

Notice that every numerator is eventually going to be cancel by the denominator, leaving the denominator containing odd powers only.So,

$$\left\{\frac{1}{1-x}\right\}\left\{\frac{1}{1-x^3}\right\}\left\{\frac{1}{1-x^5}\right\}.....$$

$$A(x) = \prod_{i=1}^{\infty}\frac{1}{(1-x^{2i-1})}$$

Since, $A(x)$ and $B(x)$ turns out to be equal we can conclude that $p^*(k) = p_o(k)$.
Hence Proved.

Q.11 Matrix multiplication is associative but not commutative.That means $AB = BA$ but $(A(BC)) = ((AB)C)$.How many ways you can multiply n matrices $A_1, A_2, A_3, ...A_n$ given in that order? You can multiply only two matrices together in your computation.

Sol.

Let us define C_{n-1} as the number of ways to parenthesize the product of n matrices $A_1.A_2.A_3....A_n$.

This problem can be seen as the parenthesize problem where P_n is number of pairs of parenthesis to be put in the valid format.

valid format is that when we traverse from left to right then at any point number of left parenthesis $\geq$ number of right parenthesis and total number of left and right parenthesis is equal.

Then $C_{n-1} = P_n$ This P_n is what we call the Catalan number.

Let us assume that we already have the counts for $0, 1, 2, 3, ..., n-1$ pairs and we would like to obtain the count for n pairs.Let p_i be the number of configurations for$i$ matching pairs of parentheses.

We know that the balanced set ,the first character has to be "(" and somewhere in the the set there is matching set ")".So in between that pair of parenthesis and right to it is other set of balanced set.$(A)B$ where A, B are balanced set of parentheses both A, B contain upto $n-1$ pairs of parentheses,but if A contains k pairs ,then B contains $n-k-1$ pairs .Here we are allowing A, B to be zero pairs .Thus we can count all configurations where A has 0 pairs and B has $n-1$ pairs,where A has 1 pairs and B has $n-2$ pairs,and so on.if we add them up,we get the total number of configurations with n balanced pairs.

$$P_n = P_{n-1}P_0 + P_{n-2}P_1 + ...P_0P_{n-1}$$

$$P_n = \sum_{i=1}^{i=n} P_{i-1}P_{n-i}$$

where $P_0 = 1$ and $P_1 = 1$. now if we replace n with $n+1$ then

$$P_{n+1} = \sum_{i=0}^{i=n} P_iP_{n-i}$$

if we try to write it as generating function,then

$$F(x) = \sum_{n\geq 0} P_n x^n$$

as we know that $p_0 = 1$ .so by applying it and taking first term out

$$F(x) = 1 + \sum_{n\geq 0} P_{n+1}x^{n+1}$$

now applying the expansion for P_{n+1}

$$F(x) = 1 + \sum_{n\geq 0}(\sum_{i=0}^{i=n} P_iP_{n-i})x^{n+1}$$

$$F(x) = 1 + x\sum_{n\geq 0}(\sum_{i=0}^{i=n}(P_ix^i)(P_{n-i})x^{n-i})$$

This inner summation is equivalent to coefficient of x^n in $F(x)^2$.

$$F(x) = 1 + x\sum_{n\geq 0}(coe.\, x^n\; in\, F(x)^2)x^n$$

$$F(x) = 1 + x(F(x)^2)$$

when solving we get $F(x) = \frac{1\pm\sqrt{1-4x}}{2x}$
now by taking the limit at $x \to 0$ then where the $F(x)$ tends to.It turns out that one with positive sign tends to infinity and one with negative sign tends to zero.So we take the negative sign.So

$$F(x) = \frac{1-\sqrt{1-4x}}{2x}$$

If we try to expand $\sqrt{1-4x}$ we get it is equal to $1 + \sum_{n\geq 0}\binom{\frac{1}{2}}{n+1}(-4x)^{n+1}$

$$F(x) = \frac{1-(1+\sum_{n\geq 0}\binom{\frac{1}{2}}{n+1}(-4x)^{n+1})}{2x}$$

$$F(x) = -\frac{1}{2x}\sum_{n\geq 0}\binom{\frac{1}{2}}{n+1}(-4x)^{n+1}$$

$$F(x) = -\frac{1}{2}\sum_{n\geq 0}\binom{\frac{1}{2}}{n+1}(-4)^{n+1}x^n$$

Now coefficient of x^n in this should give us P_n. So,

$$P_n = -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1}$$

upon simplifying it,

$$P_n = -\frac{1}{2}\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2}).....(-\frac{2n-1}{2})}{(n+1)!}(-1)^{n+1}4^{n+1}$$

$$P_n = \frac{(-1)^{2n+2}}{2^{n+2}}\frac{1.3.5......(2n-1)}{(n+1)!}4^{n+1}$$

Now if multiply and divide by $2^n(n)!$

$$P_n = \frac{1.3.5......(2n-1)}{2^{n+2}(n+1)n!}.\frac{2.4.6.8...2n}{2^n.n!}4^{n+1}$$

$$P_n = \frac{2n!.4^{n+1}}{(n+1).n!.2^n.n!.2^{n+2}}$$

$$P_n = \frac{\binom{2n}{n}4^{n+1}}{(n+1).2^{2n+2}}$$

$$P_n = \frac{\binom{2n}{n}}{(n+1)}$$

which could be written as

$$P_n = \binom{2n}{n} - \binom{2n}{n+1}$$

As we already set $C_{n-1} = P_n$ and we want the answer for C_{n-1}.So our final answer

$$C_{n-1} = \binom{2n}{n} - \binom{2n}{n+1}$$