Discrete Mathematics	Date : 7th Aug 2019

Lecture 5: Quiz 1

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1 What is the negation of the following statements:

1.1 If all rich people are happy, then all poor people are sad.

Answer :

$$\text{We know that } \neg(A \implies B) = \neg(\neg A \vee B) = A \wedge (\neg B),$$
$$\text{and } \neg(\forall x, P(x)) = \exists x, \neg P(x)$$

This statement can be written in the following manner,

$$(\forall \text{rich person, The person is happy}) \implies (\forall \text{poor person, The person is sad})$$
$$\neg(\forall \text{rich person, The person is happy} \implies \forall \text{poor person, The person is sad})$$

$\Rightarrow$ ($\forall$rich person, The person is happy) $\wedge$ $\neg$($\forall$poor person, The person is sad)

$\Rightarrow$ ($\forall$rich person, The person is happy) $\wedge$ ($\exists$poor person, $\neg$ The person is sad)

$\Rightarrow$ **All rich people are happy and *there exists* at least one poor person who is happy.**

1.2 If $G = (V, E)$ is a directed graph such that for any two vertices $u, v \in V$ at most one of (u, v) and (v, u) is in E, then, there exists a vertex $w \in V$ such that the number of vertices that are at distance 2 from w is at least the number of vertices that are at distance 1 from w.

Answer :

$$\text{We know that } \neg(A \implies B) = \neg(\neg A \vee B) = A \wedge (\neg B),$$
$$\text{and } \neg(\exists x, P(x)) = \forall x, \neg P(x)$$

This statement can be written in the following manner,

$G = (V, E)$ is a directed graph such that,

$$\left(\begin{array}{l}\text{for any two vertices } u, v \in V \text{ at most}\\ \text{one of } (u, v) \text{ and } (v, u) \in E\end{array}\right) \implies \left(\begin{array}{l}\exists w \in V,\ \#\text{ vertices at distance 2}\\ \text{from } w \geq \#\text{ vertices at distance 1}\\ \text{from } w\end{array}\right)$$

Hence the negation of the given statement is,

$$\neg\left(\begin{array}{l}\text{for any two vertices } u, v \in V \text{ at most}\\ \text{one of } (u, v) \text{ and } (v, u) \in E\end{array} \implies \begin{array}{l}\exists w \in V,\ \#\text{ vertices at distance 2}\\ \text{from } w \geq \#\text{ vertices at distance 1}\\ \text{from } w\end{array}\right)$$

$$\Rightarrow \begin{pmatrix}\text{for any two vertices } u,v \in V \text{ at most} \\ \text{one of } (u,v) \text{ and } (v,u) \in E\end{pmatrix} \wedge \neg \begin{pmatrix}\exists w \in V, \# \text{ vertices at distance 2} \\ \text{from } w \geq \# \text{ vertices at distance 1} \\ \text{from } w\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\text{for any two vertices } u,v \in V \text{ at most} \\ \text{one of } (u,v) \text{ and } (v,u) \in E\end{pmatrix} \wedge \begin{pmatrix}\forall w \in V, \# \text{ vertices at distance 2} \\ \text{from } w < \# \text{ vertices at distance 1} \\ \text{from } w\end{pmatrix}$$

So the answer is,

$G = (V, E)$ **is a directed graph such that for any two vertices** $u, v \in V$ **at most one of** (u, v) **and** (v, u) **is in** E **and for all vertex** $w \in V$**, the number of vertices that are at distance 2 from** w **is less than the number of vertices that are at distance 1 from** w**.**

2 Prove or disprove the following:

2.1 If p and q are two prime numbers then $\sqrt{pq}$ is not rational.

Ans : False.

Counter Example : Let $p = 2$ and $q = 2$. Then we have,

$$\sqrt{pq} = \sqrt{2 \times 2} = 2 \in \mathbb{Q}.$$

2.2 If w_1, w_2 and w_3 are three numbers that are not rational then $(w_1 + w_2 + w_3)$ is also not rational.

Ans : False.

Counter Example : Let $w_1 = \sqrt{2}, w_2 = \sqrt{2}, w_3 = -2\sqrt{2}$. So we have,

$$w_1 + w_2 + w_3 = \sqrt{2} + \sqrt{2} - 2\sqrt{2} = 0 \in \mathbb{Q}.$$

2.3 $(\sqrt{2} + \sqrt{3} + \sqrt{6})$ is not rational.

Ans : We will prove this by method of contradiction. Let's assume that $(\sqrt{2} + \sqrt{3} + \sqrt{6})$ is rational.

So for $p, q \in \mathbb{N}, q \neq 0, gcd(p, q) = 1$ we have,

$$\sqrt{2} + \sqrt{3} + \sqrt{6} = \frac{p}{q}$$

$$\Rightarrow \sqrt{2} + \sqrt{3} = \frac{p}{q} - \sqrt{6}$$

$$\Rightarrow (\sqrt{2} + \sqrt{3})^2 = (\frac{p}{q} - \sqrt{6})^2$$

$$\Rightarrow 5 + 2\sqrt{6} = \frac{p^2}{q^2} + \frac{2p\sqrt{6}}{q} + 6$$

$$\Rightarrow \sqrt{6}(2 - \frac{2p}{q}) = \frac{p^2}{q^2} + 1$$

$$\Rightarrow \sqrt{6} = \frac{\frac{p^2}{q^2} + 1}{2 - \frac{2p}{q}} = \frac{p^2 + q^2}{2q(q-p)}$$

We know that $\sqrt{6}$ is irrational. But

$$\frac{p^2 + q^2}{2q(q-p)} \in \mathbb{Q},$$

which leads to a contradiction.

So our assumption was incorrect, and hence $(\sqrt{2} + \sqrt{3} + \sqrt{6})$ is not rational.

3 A, B, C, D are quarreling quadruplets. If A goes to the party, then B will not go. If C goes to the party, then B will not go. Write a propositional logic statement on that would capture all set of possible combinations of A, B, C, D who may go to the party. What is the largest possible number that will go to the party?

Ans : It is Given that,

If A goes to the party, then B will not go, that means, $A \implies \neg B$

If C goes to the party, then B will not go, that means, $C \implies \neg B$

This means if any one of A or C goes then B won't go,

Since D doesn't depend on anyone, D will not be in the logic statement.

The required logic statement is, $(A \vee C) \implies \neg B$.

The largest possible number that will go is 3, i.e A, C, D.

4 Show that the propositions $(s \to r) \wedge (q \to r)$ and $(s \vee q) \to r$ are logically equivalent.

4.1 Using Truth table

s	q	r	$s \implies r$	$q \implies r$	(a) $(s \implies r) \wedge (q \implies r)$	$s \vee q$	(b) $(s \vee q) \implies r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

4.2 Using Using rules of logical equivalences

$(s \implies r) \wedge (q \implies r)$
$\Rightarrow (\neg s \vee r) \wedge (\neg q \vee r)$ (Since $A \implies B \equiv \neg A \vee B$)
$\Rightarrow (\neg s \wedge \neg q) \vee r$ (Using Distributive property)
$\Rightarrow \neg(s \vee q) \vee r$ (Using DeMorgan's law)
$\Rightarrow (s \vee q) \implies r$ (Since $\neg A \vee B \equiv A \implies B$)

5 Prove that $\forall n \in \mathbb{N}$, there exist distinct integers x, y, z for which $x^2 + y^2 + z^2 = 14^n$.(Note that $14^2 = 12^2 + 6^2 + 4^2$)

Answer : We will proove this statement by principles of mathematical induction (induct on n).

Base Case :

$P(1)$ is true, Since $14^1 = 1^2 + 2^2 + 3^2$.
$P(2)$ is true, Since $14^2 = 12^2 + 6^2 + 4^2$.

Induction Hypothesis :
Let's assume, $P(n)$ is true, *i.e.* $\exists x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 = 14^n$.
Inductive step : $P(n) \Rightarrow P(n+2)$

$14^{(n+2)} = 14^n \times 14^2 = (x^2 + y^2 + z^2)14^2$ (Since $P(n)$ is true.)
$\Rightarrow 14^{(n+2)} = (14x)^2 + (14y)^2 + (14z)^2$ $(x, y, z \in \mathbb{N} \Rightarrow 14x, 14y, 14z \in \mathbb{N})$
$\Rightarrow P(n+2)$ is true.

From this Inductive Step, we can say that since $P(1)$ is true, then $P(n)$ is true $\forall 2n+1 : n \in \mathbb{N}$

since $P(2)$ is true, then $P(n)$ is true $\forall 2n : n \in \mathbb{N}$

Hence, $P(n)$ is true $\forall n \in \mathbb{N}$.

6 Prove that a graph is bipartite if and only if the graph has no odd cycle.

Answer :

6.1 Graph is bipartite $\implies$ Graph has no odd cycle

Let G be a bipartite graph with vertex set V_1 and V_2. One step will always take us from V_1 to V_2 or V_2 to V_1. So to complete a cycle(or get back to where we start) we need even number of steps. Hence we can conclude that there is no odd cycle.

6.2 Graph has no odd cycle $\implies$ Graph is bipartite

Suppose that G has no odd cycle. Let us choose any vertex $v \in G$. We can construct two sets of vertices say V_1 and V_2 such that,

V_1 is the set of vertices such that the shortest path from each element of V_1 to v is of odd length.

V_2 is the set of vertices such that the shortest path from each element of V_1 to v is of even length.

Clearly V_1 and V_2 are partitions V.

Suppose there exists a vertex $u \in V_1$ such that the path from v from u is of even length. Since V_1 is the collection of vertices that are at odd distance from v, we can find an odd cycle from v passing through u, which gives a condradiction. Hence we can conclude

$$V_1 \cap V_2 = \phi.$$

No two vertices in V_1 or in V_2 are adjacent: otherwise an odd cycle is induced in G. So G is bipartite.

Hence a graph is bipartite iff the graph has no odd cycle.