Discrete Mathematics	DATE : 21 Oct 2019

Quiz 2: Solutions

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1. Let $a_0 = 1$ and $a_1 = 2$ and for all $n \geq 0$ we have $a_{n+2} = 5a_{n+1}$ - $4a_n$. Using generating functions find a_n in terms of n.

Solution:

Let $F(x)$ be generating function for the given recurrence relation, then

$$\begin{aligned} F(x) &= \sum_{i=0}^{\infty} a_i x^i \\ &= a_0 + a_1 + \sum_{i=2}^{\infty} a_i x^i \\ &= 1 + 2x + \sum_{i=2}^{\infty}(5a_{i-1} - 4a_{i-2})x^i \\ &= 1 + 2x + 5x\sum_{i=2}^{\infty} a_{i-1}x^{i-1} - 4x^2\sum_{i=2}^{\infty} a_{i-2}x^{i-2} \\ &= 1 + 2x - 5x + 5x(1 + \sum_{i=1}^{\infty} a_i x^i) - 4x^2 \sum_{i=0}^{\infty} a_i x^i\text{f} \\ &= 1 - 3x + 5x\text{F(x)} - 4x^2\text{F(x)} \end{aligned}$$

$$F(x) = \frac{1 - 3x}{4x^2 - 5x + 1}$$

$$F(x) = \frac{1 - 3x}{(1 - 4x)(1 - x)}$$

Using partial fraction to simplify the following function, we get

$$F(x) = \frac{1}{3(1 - 4x)} + \frac{2}{3(1 - x)}$$

$$F(x) = \frac{1}{3}\sum_{k=0}^{\infty} 4^k x^k + \frac{2}{3}\sum_{k=0}^{\infty} x^k$$

We need to find the coefficient of x^n in the above function which is equal to a^n,

$$a^n = \frac{1}{3}4^n + \frac{2}{3}$$

$$a^n = \frac{4^n + 2}{3}$$

2.How many n digits integers are there where no 0 does not follow 8 immediately?

Solution:

Let consider T_n be the number of integers of size n where no 0 follows 8 immediately. To form the recurrence relation of T_n, consider following diagram,

1	2	.	.	.	n-1	n
b_1	b_2	b_-	b_-	b_-	b_{n-1}	b_n

If we consider b_n digit to be non-zero, then there are 9 possible choices for b_n, giving $9T_{n-1}$ term.
If the b_n digit is 0, then again 2 cases,
If b_{n-1} is non zero, so 8 choices for the b_{n-1} digit, giving $8T_{n-2}$.
If b_{n-1} digit is zero, giving $8T_{n-3}$.
Repeat this way until 1 digit is left which has 8 possibilities (1 to 9 except 8) This gives the following recurrence,

$$T_n = 9T_{n-1} + 8T_{n-2} + 8T_{n-3} + ... + 8$$

Similarly, we can write recurrence for T_{n-1} as,

$$T_{n-1} = 9T_{n-2} + 8T_{n-3} + 8T_{n-4} + ... + 8$$

$$T_n - T_{n-1} = 9T_{n-1} - T_{n-2}$$

(since we have written T_{n-1} as well in terms of smaller ones)

$$T_n = 10T_{n-1} - T_{n-2} \text{ , } n \geq 4$$

where $T_1 = 9, T_2 = 88, T_3 = 872$, calculated using the above approach.

Solving the above recurrence using generating functions. Let $F(x)$ be generating function for T_n, then

$$\begin{aligned} F(x) &= \sum_{i=1}^{\infty} T_i x^i \\ &= T_1 x + T_2 x^2 + T_3 x^3 + \sum_{i=4}^{\infty} T_i x^i \\ &= 9x + 88x^2 + 872x^3 + \sum_{i=4}^{\infty} (10T_{i-1}x^i - T_{i-2}x^i) \\ &= 9x + 88x^2 + 872x^3 + 10x \sum_{i=3}^{\infty} T_i x^i - x^2 \sum_{i=2}^{\infty} T_i x^i) \\ &= 9x + 88x^2 + 872x^3 + 10x(F(x) - 9x - 88x^2) - x^2(F(x) - 9x) \\ &= 9x - 2x^2 + x^3 + 10xF(x) - x^2 F(x) \end{aligned}$$

$$F(x) = \frac{x^3 - 2x^2 + 9x}{x^2 - 10x + 1}$$

$$F(x) = x\frac{x^2 - 2x + 9}{x^2 - 10x + 1}$$

If we take x on other side, then we have to find the coefficient of $n-1$ in F_x.

$$T_n = coef.of x^{n-1} in \left(\frac{x^2 - 2x + 9}{x^2 - 10x + 1} \right)$$

$$T_n = coef.of x^{n-1} in \left(1 + \frac{8x+8}{x^2-10x+1}\right)$$

Let assume roots of the equation $x^2 - 10x + 1$ be α and β, then

$$T_n = coef.of x^{n-1} in \left(8\frac{x+1}{(x-\alpha)(x-\beta)}\right)$$

$$T_n = coef.of x^{n-1} in \left(8\frac{x-\alpha+1+\alpha}{(x-\alpha)(x-\beta)}\right)$$

$$T_n = coef.of x^{n-1} in \left(\frac{8}{x-\beta} + \frac{8(1+\alpha)}{\alpha-\beta}\frac{\alpha-\beta}{(x-\alpha)(x-\beta)}\right)$$

$$T_n = coef.of x^{n-1} in \left(\frac{8}{x-\beta} + \frac{8(1+\alpha)}{\alpha-\beta}\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right)\right)$$

$$T_n = coef.of x^{n-1} in \left(\frac{8}{x-\beta} + \frac{8(1+\alpha)}{\alpha-\beta}\frac{1}{x-\alpha} - \frac{8(1+\alpha)}{\alpha-\beta}\frac{1}{x-\beta}\right)$$

$$T_n = coef.of x^{n-1} in \left(\frac{8(1+\alpha)}{\alpha-\beta}\frac{1}{x-\alpha} + \left(8 - \frac{8(1+\alpha)}{\alpha-\beta}\right)\frac{1}{x-\beta}\right)$$

Take constants

$$A = \frac{8(1+\alpha)}{\alpha-\beta}$$

and

$$B = 8 - \frac{8(1+\alpha)}{\alpha-\beta}$$

, then

$$T_n = coef.of x^{n-1} in \left(A\frac{1}{x-\alpha} + B\frac{1}{x-\beta}\right)$$

3. (a) Let us consider the following sequence of numbers:

$$a_n = \begin{cases} 1; & 0 \le n \le 3. \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}; & n \le 4\ . \end{cases} \tag{1}$$

Prove that $a_n \equiv 1 (\text{mod } 3)$.

Solution:

We will prove this using strong induction.
We induct on n.
Base case: For n = 0, 1, 2, 3, 4,
$a_0 = a_1 = a_2 = a_3 = 1 \equiv 1(mod 3)]$
and,

$$a_4 = a_0 + a_1 + a_2 + a_3$$
$$a_4 = 4 \equiv 1(mod 3)$$

Induction Hypothesis: Let us assume this to be true for $n =$ 0,1,2,3.....k. i.e.

$$a_i \equiv 1(mod 3) \text{ for } 0 \leq i \leq k$$

Induction Step: We will show that this is true for $n =$ k+1. i.e. $(a_k, a_{k-1}, a_{k-2}, a_{k-3}) \implies a_{k+1}$

$$a_{k+1} = a_k + a_{k-1} + a_{k-2} + a_{k-3}$$
$$a_{k+1} = 1(mod 3)+1(mod 3)+1(mod 3)+1(mod 3) \qquad (\because hypothesis)$$
$$a_{k+1} \equiv 4(mod 3) \equiv 1(mod 3)$$

Hence, Proved that $a_n \equiv 1$(mod 3).

(b) How many integer solutions of the equation $x + y + z =$ 6 satisfy -1 $\leq x \leq$ 2, 1 $\leq y, z \leq$ 4?

Solution:

Generating function for the given problem will look like,

$$F(x) = (x^{-1} + x^0 + x^1 + x^2)(x^1 + x^2 + x^3 + x^4)(x^1 + x^2 + x^3 + x^4)$$

$$= (x^{-1} + 1 + x + x^2)(x + x^2 + x^3 + x^4)^2$$
$$= (x^{-1} + 1 + x + x^2)x^2(1 + x^1 + x^2 + x^3)^2$$
$$= x(x + x^2 + x^3 + x^4)^3$$

Therefore, Number of solutions for the given equation is equals to

$$= \text{coef. of } x^6 \text{ in } [x(x + x^2 + x^3 + x^4)^3]$$
$$= \text{coef. of } x^5 \text{ in } [(x + x^2 + x^3 + x^4)^3]$$
$$= \text{coef. of } x^5 \text{ in } \left[\left(\frac{1-x^4}{1-x}\right)^3\right]$$
$$= \text{coef. of } x^5 \text{ in } [(1 - x^4)^3(1 - x)^{-3}]$$
$$= \text{coef. of } x^5 \text{ in}$$
$$\left[\left(1 - \binom{3}{1}x^4 + \binom{3}{2}x^8 - \binom{3}{3}x^{12}\right)\left(1 + \binom{-3}{1}x + \binom{-3}{2}x^2 + \binom{-3}{3}x^3 + ...\right)\right]$$

$= \begin{pmatrix} -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3+5-1 \\ 5 \end{pmatrix} - 9 \qquad \because \begin{pmatrix} -n \\ r \end{pmatrix} = \begin{pmatrix} n+r-1 \\ r \end{pmatrix}$

= 21 - 9

= 12

4. Let $a_0 = 1$ and $a_1 = 5$ and for all $n \geq 0$ we have $a_{n+2} = a_{n+1}$ - 6a_n. Using generating functions find a_n in terms of n.

Solution:

Let $F(x)$ be generating function for the given recurrence relation, then

$F(x) = \sum_{i=0}^{\infty} a_i x^i$
$= a_0 + a_1 + \sum_{i=2}^{\infty} a_i x^i$
$= 1 + 5x + \sum_{i=2}^{\infty}(a_{i-1} - 6a_{i-2})x^i$
$= 1 + 5x + x\sum_{i=1}^{\infty} a_i x^i - 6x^2 \sum_{i=0}^{\infty} a_i x^i$
$= 1 + 4x + x(1 + \sum_{i=1}^{\infty} a_i x^i) - 6x^2 F(x)$
$= 1 + 4x + xF(x) - 6x^2 F(x)$

$$F(x) = \frac{1+4x}{6x^2 - x + 1}$$

Here the equation $6x^2 - x + 1$ have two complex roots, since $b^2 - 4ac$ = 1 - 4(6) = -23. So we assume these roots to be α and β, where

$$\alpha = a_1 i + b_1 \text{ and } \beta = a_2 i + b_2$$

Then,

$$F(x) = \frac{1+4x}{(x-\alpha)(x-\beta)}$$

Therefore, a_n will be coefficient of x^n in $F(x)$, i.e.

$$a_n = coeff.of x^n in \left[\frac{1+4x}{(x-\alpha)(x-\beta)}\right]$$

5. (a) Prove that for every positive integer n, there exist at least n consecutive composite numbers.

Solution:

A composite number is one for which we can write $n = ab$ for some $1 < a < n$, and $1 < b < n$. Equivalently, n is composite if there is some number $1 < a < n$ such that $a|n$.

Consider, as suggested, the numbers

2 + (n + 1)!, 3 + (n + 1)!, . . . , n + (n + 1)!,(n + 1) + (n + 1)!.

Note that there are n of these numbers. The goal is to show that these are all composite.

As an example, consider $n = 4$. Then $(4 + 1)! = 5! = 120$, and so these numbers are

122, 123, 124, 125

The first is even, the second, divisible by three, the fourth is even, and the last divisible by 5. So they are all composite.

We first note that by definition, we always have that if $k \leq m$, then $k|m!$. Thus for all integers $2 \leq k \leq (n + 1)$, we have that $k|(n + 1)!$. We also have that for all such integers, $k|k$ trivially.

Lastly, we have that if $k|a$ and $k|b$, that $k|(a + b)$.
It now follows that

2 — 2 + (n + 1)!
3 — 3 + (n + 1)!
.
.
.
n — n + (n + 1)!
n + 1 — n + 1 + (n + 1)!

and so all of these numbers, having proper divisors, must be composite.

(b) A codeword is a 6 digit code (composed of digits 0-9) containing at least one of each of the digits 0,1,2,3 and 4. How many such codewords are there? Examples: 012345, 932410 etc.

Solution:

Consider the codeword of 6 digits,

1	2	3	4	5	6
b_1	b_2	b_3	b_4	b_5	b_6

Digits 0-4 must be present in the code, so we have to choose one more digit to be there in the code and compute their arrangements.

It can be done in two ways, Suppose we choose the 6th digit to be different from 0-4 digits, then number of codewords formed with these digits equals

$$\binom{5}{1} 6! = 5.6!$$

If we choose the 6th digit as one of the 0-4 digits, then number of codewords formed with these digits equals

$$\binom{5}{1} \frac{6!}{2!} = 5.\frac{6!}{2}$$

Hence, The total number of codewords formed with the given constrains equals to

$$5.6! + 5.\frac{6!}{2!} = 5.720 + 5.360$$
$$= 5400$$

References

https://wiki.ubc.ca/images/1/10/Soln2.pdf