Discrete Maths 30.10.19

Solution: Quiz 3

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1 Solution 1:

Suppose the given graph G is planar.
Let no. of vertices be $|V|$, no. of edges be $|E|$ and no. of faces be $|F|$.
Then, for any planar graph:
$|V|$ -$|E|$+$|F|$=2. ...(i)
Let $r_1, r_2, ..r_{|F|}$ *be no.of edges*
in Faces $f1, f2..f_{|F|}$ *respectively.*
$\sum_{i=0}^{|F|} r_i = 2|E|$
Here, $r_i = 5$
This implies, $5|F|=2|E|$
This implies, $|F|=2/5|E|$
Putting value of $|F|$ in equation in (i)
This implies, $|E|=5/3(|V|-2)$
Here, $|V|=10$, $|E|=15$
On putting these values, L.H.S=15 but R.H.S=40/3.
So, L.H.S $\neq$ *R.H.S*
Hence, given graph is not a planar graph.

2 Solution 2:

Let no. of vertices be $|V|$, no. of edges be $|E|$ and no. of faces be $|F|$.
Then, for any planar graph:
$|V|-|E|+|F|=2$. ...(i)
Let $r_1, r_2..r_F$ *be no.of edges in Faces* $f1, f2..f_{|F|}$*respectively.*
$\sum_{i=0}^{|F|} r_i = 2|E|$
$Here, r_i >= 6$
$Thisimplies, 2|E| \geq 6|F|$
$or, |F| \leq |E|/3$
$This\ implies, |V| - |E| + |F| \leq |V| - 2/3|E|$
$This\ implies,\ 2|E| \leq 3(|V| - 2)$
Let d_i *be the degree of vertex* v_i.
$Then, \sum_{i=0}^{|V|} d_i = 2|E|$
$This\ implies, \sum_{i=0}^{|V|} d_i \leq 3(|V| - 2)..(ii)$
$Suppose,\ each\ d_i \geq 3$
$This\ implies, \sum_{i=0}^{|V|} d_i \geq 3|V|$
But, this contradicts equation (ii)

This means there exist a vertex v with degree less than or equal to 2.
Now, we are applying induction on no. of vertices.
Base Case: If there are less than or equal to 3 vertices
then then graph is 3 vertex colorable.
Induction Hypothesis: Suppose any planar graph G with k vertices with girth greater than or equal to 6 is 3 vertex colorable.
Induction step: Consider a planar graph
G' with k+1 vertices and with girth greater than or equal to 6.
For this graph there must exist a vertex
v whose degree is less than or equal to 2.
Now, G'/v will also be planar and still with girth greater than or equal to 6.
By using Induction Hypothesis, G'/v is a k vertices planar graph with girth greater than or equal to 6.
Now, if we add vertex v in G'/v then it will have at most 2 neighbours.
So, we can color this vertex v with color
different from its neighbour. Hence this makes G a 3 vertex colorable.