

Snakes and perturbed random walks

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Abstract

In this paper we study some properties of random walks perturbed at extrema, which are generalizations of the walks considered e.g. in Davis (1999). This process can also be viewed as a version of *excited random walk*, studied recently by many authors. We obtain a few properties related to the range of the process with infinite memory. We also prove the strong law, CLT, and the criterion for the recurrence of the perturbed walk with finite memory.

Key words: excited / perturbed / cookie random walk, recurrence, transience.

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1 Introduction

This paper has been inspired by the results of Davis (1999) for the random walks perturbed at extrema. Davis (1999) studied the stochastic process which is the limit one of the walks described below. Our purpose here is to study the properties of these perturbed walks as they are, without considering the limit process found in Davis (1999). Let $0 < p < 1$ and $0 < q < 1$. Fix a number L which is either a positive integer or $+\infty$. We define a *perturbed at L -extrema random walk* X_k as a nearest-neighbour random walk on \mathbb{Z}^1 with the transitional probabilities $\mathbb{P}(X_{k+1} = x + 1 | X_k = x) = 1 - \mathbb{P}(X_{k+1} = x - 1 | X_k = x)$ equal to

$$\begin{cases} p, & \text{if } x = \max_{m=0,1,\dots,L\wedge k} X_{k-m}; \\ q, & \text{if } x = \min_{m=0,1,\dots,L\wedge k} X_{k-m}; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

For definiteness, if x is both the maximum and the minimum, we let this probability be $\frac{1}{2}$ (this obviously happens only when $k = 0$).

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When L is finite, we will call this walk *a walk with finite memory*. In this case, it is natural to think of this process as of a “snake” (hence the title of the paper, compare our process with a famous video game, released during the mid 1970s, see Surhone, Tennoe, and Henssonow (2010)) of length L units moving on the integers, whose transition probabilities depend on whether the snake is surrounded by parts of its body or not.

The case $L = \infty$ corresponds to the walk perturbed at global extrema, and we will refer to this walk as the walk with *infinite memory*. It was shown in Davis (1999) that this walk, properly rescaled, converges to a certain stochastic process. Other relevant papers are Davis (1990, 1996); Benjamini and Wilson (2003) and Volkov (2003) studied an excited random walk (ERW), which transitional probabilities differ when the walk visits a site for the first time. Zerner (2005) studied multi-excited random walks on integers. More recently, Basdevant and Singh (2008a) got some interesting results on the speed of ERW, confirming certain conjectures posed in Zerner (2005), and computed the exact rate of growth in the zero-speed regime in (2008b), while Kosygina and Zerner (2008) obtained annealed CLT for ERW, using branching theory techniques.

When $L = \infty$, our process can be viewed as a special ERW as follows. Consider a site x on a positive axis, and place a geometric number M_x of cookies at the site, such that $\mathbb{P}(M_x = k) = p(1 - p)^{k-1}$, $k = 1, 2, \dots$, and M_x , $x = 1, 2, \dots$, are i.i.d. random variables. Whenever the walk visits site x and there are at least two cookie there, it eats one cookie and goes to the left. When it eats the last cookie, it goes to the right. Finally, when there are no cookies left at a site, the walk goes left or right with equal probabilities. On the negative axis, the process is defined similarly with the number of the cookies there distributed according to the law $\mathbb{P}(M_x = k) = (1 - q)q^{k-1}$, $x = -1, -2, \dots$. There are no cookies at site 0.

Note that in Kosygina and Zerner (2008) the number of cookies is uniformly bounded, which is not the case in our model; also our model resembles *drilling random walk* introduced in Volkov (2003).

In Section 2, for $L = \infty$ we focus on the behaviour of the process in *finite time*, as opposed to Davis (1999). In particular, we obtain some results on the time it takes before the length of the visited (“cookie-free”) area reaches a certain value. We also compute the limiting probability to be at the global maxima, given that the walk is at one of its extrema.

In Section 3, for $L < \infty$, the process does not resemble ERW mentioned above, and can be viewed as a Markov chain on a product space of $\{-1, +1\}^L \times \mathbb{Z}$. We establish that the finite memory walk is recurrent if and only if $p + q = 1$ (compare this with the case $L = \infty$ when the walk is always recurrent). Also we obtain strong law and CLT for our process, and show how the speed of the transient process, when $p + q \neq 1$, decays as L grows.

We conclude with conjecture and open problems in Section 4.

Before we proceed with the next section, let us introduce a few notations. Let

$$\begin{aligned}\mathcal{M} &= \mathcal{M}(k) = \mathcal{M}_L(k) = \max_{i=1,2,\dots,L \wedge k} X_{k+1-i}, \\ \mathbf{m} &= \mathbf{m}(k) = \mathbf{m}_L(k) = \min_{i=1,2,\dots,L \wedge k} X_{k+1-i}.\end{aligned}$$

If $X_k \neq \mathbf{m}$ and $X_k \neq \mathcal{M}$ then our process behaves exactly as a simple random walk (SRW). The only differences between X_k and the SRW occur when either $X_k = \mathbf{m}$ or $X_k = \mathcal{M}$. Let $\mathcal{R}(k) = \mathcal{M}(k) - \mathbf{m}(k)$ be the range (the spread) of the walk and $\rho_n = \min\{k : \mathcal{R}(k) = n\}$ be the stopping time when this range reaches n . At some point we will be interested in the quantity $\Delta_{n,n+1} = \rho_{n+1} - \rho_n$.

These notations \mathcal{M} , \mathbf{m} , ρ_n , and $\Delta_{n,n+1}$ will be used throughout the paper.

2 Properties of the perturbed random walks with infinite memory: $L = +\infty$

In this section we establish some interesting properties of the perturbed random walks with infinite memory.

To characterize its distribution, without loss of generality suppose that $\mathbf{m}(k) = 0$, $\mathcal{M}(k) = n$, and let $\xi_l^{(n)} = \inf\{t : X_t = -1 \text{ or } n+1 \mid X_0 = l\}$ be the random time until the range is increased given that the walk starts at l , $0 \leq l \leq n$. Since $\phi_l = \phi_l^{(n)}(\lambda) = \mathbb{E} \exp(-\lambda \xi_l^{(n)})$, the Laplace transform of $\xi_l^{(n)}$ satisfies

$$\begin{aligned}\phi_0 &= e^{-\lambda}[q\phi_1 + (1-q) \cdot 1], \quad l = 0; \\ \phi_l &= e^{-\lambda}[0.5\phi_{l-1} + 0.5\phi_{l+1}], \quad 0 < l < n; \\ \phi_n &= e^{-\lambda}[p \cdot 1 + (1-p)\phi_{n-1}], \quad l = n,\end{aligned}\tag{2.1}$$

it can be represented for $\lambda \geq 0$ as

$$\phi_l = C_1 \zeta^l + C_2 \zeta^{-l}, \quad \text{where } \zeta = e^\lambda + \sqrt{e^{2\lambda} - 1}.$$

The coefficients C_1, C_2 should be chosen to satisfy (2.1). Solving for them, we obtain that

$$\phi_0 = \frac{(1-q)e^\lambda \zeta^2 (1 - \zeta^{2n}) - (1-p)(1-q)\zeta^3 (1 - \zeta^{2n-2}) + pq\zeta^{n+1}(1 - \zeta^2)}{e^{2\lambda} \zeta^2 (1 - \zeta^{2n}) - (1+q-p)e^\lambda \zeta^3 (1 - \zeta^{2n-2}) + q(1-p)\zeta^4 (1 - \zeta^{2n-4})}$$

and

$$\phi_n = \frac{p\zeta^2 e^\lambda (1 - \zeta^{2n}) - pq\zeta^3 (1 - \zeta^{2n-2}) + (1-p)(1-q)\zeta^{n+1}(1 - \zeta^2)}{e^{2\lambda} \zeta^2 (1 - \zeta^{2n}) - (1+q-p)e^\lambda \zeta^3 (1 - \zeta^{2n-2}) + q(1-p)\zeta^4 (1 - \zeta^{2n-4})}$$

Let $D_{\mathcal{M}} = D_{\mathcal{M}}(n) = \mathbb{E}(\Delta_{n,n+1} | X_{\rho_n} = \mathcal{M}(\rho_n))$ and $D_{\mathbf{m}} = D_{\mathbf{m}}(n) = \mathbb{E}(\Delta_{n,n+1} | X_{\rho_n} = \mathbf{m}(\rho_n))$. Then

$$\begin{aligned} D_{\mathcal{M}} - \left. \frac{d\phi_n}{d\lambda} \right|_{\lambda=0} &= \frac{(1-p-q+pq)n^2 + (p+2q-3pq)n + 1-p-q+2pq}{1-q-p+2pq+p(1-q)n} \\ D_{\mathbf{m}} - \left. \frac{d\phi_0}{d\lambda} \right|_{\lambda=0} &= \frac{pqn^2 + (p+2q-3pq)n + 1-p-q+2pq}{1-q-p+2pq+p(1-q)n}. \end{aligned}$$

Note that for large n we have

$$D_{\mathcal{M}}(n) = \frac{1-p}{p}n + O(1), \quad D_{\mathbf{m}}(n) = \frac{q}{1-q}n + O(1). \quad (2.2)$$

Next, let $p_{\mathcal{M}} = p_{\mathcal{M}}(n) = \mathbb{P}(X_{\rho_{n+1}} = \mathcal{M}(\rho_{n+1}) | X_{\rho_n} = \mathcal{M}(\rho_n))$ and $p_{\mathbf{m}} = p_{\mathbf{m}}(n) = \mathbb{P}(X_{\rho_{n+1}} = \mathbf{m}(\rho_{n+1}) | X_{\rho_n} = \mathbf{m}(\rho_n))$ be the probabilities that once the range has increased at the maximum (resp. minimum) the next increase will take place again at the maximum (resp. minimum). Then these probabilities satisfy

$$\begin{aligned} p_{\mathcal{M}} &= p + (1-p) \left(\frac{n-1}{n} p_{\mathcal{M}} + \frac{1}{n} (1-p_{\mathbf{m}}) \right), \\ p_{\mathbf{m}} &= 1-q + q \left(\frac{n-1}{n} p_{\mathbf{m}} + \frac{1}{n} (1-p_{\mathcal{M}}) \right). \end{aligned}$$

The solution to this system is

$$\begin{aligned} p_{\mathcal{M}} = p_{\mathcal{M}}(n) &= \frac{p(1-q)n + pq}{1-p-q+2pq+p(1-q)n}, \\ p_{\mathbf{m}} = p_{\mathbf{m}}(n) &= \frac{p(1-q)n + pq - p - q + 1}{1-p-q+2pq+p(1-q)n}. \end{aligned} \quad (2.3)$$

For large n both probabilities are close to one:

$$p_{\mathcal{M}}(n) = 1 - \frac{1-p}{p} \times \frac{1}{n} + O(n^{-2}), \quad p_{\mathbf{m}}(n) = 1 - \frac{q}{1-q} \times \frac{1}{n} + O(n^{-2}).$$

Now consider the induced chain Y_n with $Y_n = 1$ if $X_{\rho_n} = \mathcal{M}(\rho_n)$, and $Y_n = 0$ if $X_{\rho_n} = \mathbf{m}(\rho_n)$. It is straightforward that Y_k is a time-nonhomogeneous Markov chain. We state, however, that it still has a limiting distribution:

Proposition 1

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n = 1) = \frac{pq}{1-p-q+2pq} =: \pi_{\mathcal{M}}.$$

Proof: The matrix of transitional probabilities for the chain $\{Y_n\}$ with the states $\{1, 0\}$ is

$$A_n = \begin{pmatrix} p_{\mathcal{M}}(n) & 1-p_{\mathcal{M}}(n) \\ 1-p_{\mathbf{m}}(n) & p_{\mathbf{m}}(n) \end{pmatrix} = \begin{pmatrix} 1-a/n & a/n \\ b/n & 1-b/n \end{pmatrix} + O(n^{-2})$$

where $a = (1 - p)/p$, $b = q/(1 - q)$, using the formulas for $p_{\mathcal{M}}(n)$ and $p_{\mathfrak{m}}(n)$ given by (2.3).

Observe that

$$A_n = BC_n B^{-1} + O(n^{-2})$$

where

$$B = \begin{pmatrix} 1 & -a/b \\ 1 & 1 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 - (a + b)/n \end{pmatrix}$$

Hence, for $m \geq n$,

$$\begin{aligned} A_n A_{n+1} A_{n+2} \cdots A_m &= B \begin{pmatrix} 1 & 0 \\ 0 & \prod_{k=n}^m (1 - \frac{a+b}{k}) \end{pmatrix} B^{-1} + O(n^{-1} - m^{-1}) \\ &= \frac{1}{a+b} \begin{pmatrix} b + a(n/m)^{a+b} & a - a(n/m)^{a+b} \\ b - b(n/m)^{a+b} & a + b(n/m)^{a+b} \end{pmatrix} + O((m - n)n^{-2}) \end{aligned} \quad (2.4)$$

In particular, if $\pi_n = \mathbb{P}(Y_n = 1) = b/(a + b) + \delta_n = \pi_{\mathcal{M}} + \delta_n$, then

$$\pi_{n+1} = \mathbb{P}(Y_{n+1} = 1) = \pi_n p_{\mathcal{M}}(n) + (1 - \pi_n)(1 - p_{\mathfrak{m}}(n)) = \pi_{\mathcal{M}} + \delta_{n+1}$$

where

$$\delta_{n+1} = \left(1 - \frac{a+b}{n} + O(n^{-2})\right) \delta_n. \quad (2.5)$$

Since the sequence δ_n 's satisfies (2.5) and $a + b > 0$, it is easy to see that $\lim_{n \rightarrow \infty} \delta_n = 0$ and therefore $\lim_{n \rightarrow \infty} \pi_n = \pi_{\mathcal{M}}$. (In fact, we can even conclude that $\pi_n = \pi_{\mathcal{M}} + O(1)/n^{a+b}$). \blacksquare

Theorem 1 (a) For large n

$$\mathbb{E} \Delta_{n,n+1} = \frac{q(1-p)n}{1-p-q+2pq} + o(n).$$

(b) Asymptotically,

$$\mathbb{E} \rho_n = \frac{q(1-p)}{1-p-q+2pq} \times \frac{n^2}{2} + o(n^2).$$

Proof: We start with part (a). Observe that

$$\begin{aligned}\mathbb{E} \Delta_{n,n+1}/n &= \frac{\mathbb{E}(\Delta_{n,n+1} | X_{\rho_n} = \mathcal{M}(\rho_n))}{n} \mathbb{P}(Y_n = 1) \\ &+ \frac{\mathbb{E}(\Delta_{n,n+1} | X_{\rho_n} = \mathbf{m}(\rho_n))}{n} \mathbb{P}(Y_n = 0) \\ &= \frac{D_{\mathcal{M}}(n)}{n} \pi_n + \frac{D_{\mathbf{m}}(n)}{n} (1 - \pi_n) \longrightarrow \frac{1-p}{p} \pi_{\mathcal{M}} + \frac{q}{1-q} (1 - \pi_{\mathcal{M}})\end{aligned}$$

by Proposition 1 and formula (2.2). Now part (b) of the corollary immediately follows from part (a) and the fact that $\rho_n = \sum_{i=0}^{n-1} \Delta_{i,i+1}$. \blacksquare

3 Finite memory: $0 < L < \infty$

Note that if we include the history of the process X_k for the past L steps, it becomes a Markov chain. Formally, let $Y_k = (\eta_k^1, \eta_k^2, \dots, \eta_k^L)$ be the sequence of -1 's or $+1$'s of length L , with $\eta_k^i := X_{k-i+1} - X_{k-i}$, $i = 1, 2, \dots, L$. From Y_k 's it is possible to extract the information whether the process X_k hit its local maximum or minimum, as described below. Therefore, the pair (X_k, Y_k) is a Markov chain; moreover Y_k 's itself form a Markov chain on the space $\mathfrak{E}^{(L)} = \{-1, +1\}^L$ of the sequences of plus and minus ones of length L .

If $Y_k = \eta = (\eta^1, \eta^2, \dots, \eta^L)$, then let $S(\eta, j)$ denote $\eta^1 + \eta^2 + \dots + \eta^j$, $j = 1, 2, \dots, L$. We say that η is a local maximum, if $S(\eta, j) \geq 0$ for $j = 1, 2, \dots, L$; local minimum if $S(\eta, j) \leq 0$ for $j = 1, 2, \dots, L$; and “neither” otherwise. Then X_k is a local maximum (minimum resp.) if and only if Y_k is a local maximum (minimum resp.)

Note that from each of the 2^L states of $\mathfrak{E}^{(L)}$ $Y_k = (\eta^1, \eta^2, \dots, \eta^L)$ can go only to two states: $Y_{k+1} = (\eta_*, \eta^1, \eta^2, \dots, \eta^{L-1})$, where $\eta_* = +1$ or -1 . Observe also that

$$\mathbb{P}(\eta_* = +1) = 1 - \mathbb{P}(\eta_* = -1) = \begin{cases} p & \text{if } Y \text{ is a local maximum;} \\ q & \text{if } Y \text{ is a local minimum;} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Since the space $\mathfrak{E}^{(L)}$ is finite, and Y_k is obviously irreducible, there exists the limiting occupational measure for Y_k denoted as $\pi = \{\pi^{(L)}(\eta)\}_{\eta \in \mathfrak{E}^{(L)}}$, which obviously depends on p , q , and L .

Let

$$\pi(\max) = \sum_{\eta \text{ is local max.}} \pi(\eta)$$

and

$$\pi(\min) = \sum_{\eta \text{ is local min.}} \pi(\eta)$$

Observe that $0 < \pi(\min) < 1$ and $0 < \pi(\max) < 1$, and set $\Delta = \Delta_L := (2p - 1)\pi(\max) + (2q - 1)\pi(\min)$.

Lemma 1

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = \Delta_L \text{ a.s.}$$

and hence if $\Delta_L \neq 0$ then X_n is transient.

Proof of the Lemma Let $f : \mathfrak{E}^{(L)} \rightarrow \{-1, +1\}$ be such that $f(\eta) = \eta^1$, i.e. the first coordinate of η . By the strong law for the Markov chains (see e.g. [4], p.145)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=1}^n f(Y_m) = \mathbb{E}_\pi f(Y_1)$$

Since by the construction Y_k 's we have $\sum_{m=1}^n f(Y_m) = \sum_{m=1}^n (X_m - X_{m-1}) = X_n - X_0$, it suffices to show $\mathbb{E}_\pi f(Y_1) = \Delta$.

Indeed,

$$\begin{aligned} \mathbb{E}_\pi f(Y_1) = \mathbb{E}_\pi Y_1^1 &= \pi(\max) \mathbb{E}_\pi(Y_1^1 | Y_0 \text{ is loc.max.}) \\ &+ \pi(\min) \mathbb{E}_\pi(Y_1^1 | Y_0 \text{ is loc.min.}) \\ &+ (1 - \pi(\max) - \pi(\min)) \mathbb{E}_\pi(Y_1^1 | Y_0 \text{ is neither}) \\ &= \pi(\max)(2p - 1) + \pi(\min)(2q - 1) \\ &+ (1 - \pi(\max) - \pi(\min)) \times 0 = \Delta. \end{aligned}$$

The last statement of the Lemma is straightforward. ■

Lemma 2 $\Delta < 0, = 0, \text{ or } > 0$ if $p + q - 1 < 0, = 0, \text{ or } > 0$ respectively.

Proof: Case 1: $p + q = 1$. In this case, the chain Y_n is symmetric. Indeed, if we replace each $+1$ by -1 and vice versa, it will have the same distribution since $q = 1 - p$. Thus by symmetry $\pi(\max) = \pi(\min)$. whence $\Delta = (2p - 1)\pi(\max) + (2q - 1)\pi(\min) = ((2p - 1) + (2q - 1))\pi(\max) = 2(p + q - 1)\pi(\max) = 0$.

Case 2a: $p + q > 1, p, q \geq \frac{1}{2}$. In this case, $2p - 1 \geq 0$ and $2q - 1 > 0$ or $2p - 1 > 0$ and $2q - 1 \geq 0$. Hence, $\Delta = (2p - 1)\pi(\max) + (2q - 1)\pi(\min) > 0$

Case 2b: $p + q > 1, p \geq \frac{1}{2} \geq q$. Construct a new *symmetric* chain $\{\tilde{Y}_n\}$ with $\tilde{p} = p$ and $\tilde{q} = 1 - \tilde{p} < q \leq \frac{1}{2}$. Denote π and $\tilde{\pi}$ the stationary distributions for Y_n and \tilde{Y}_n respectively. Using a sequence of i.i.d. *Uniform* $[0, 1]$ random variables U_1, U_2, \dots we will use couple Y_n and \tilde{Y}_n to demonstrate that $\pi(\max) \geq \tilde{\pi}(\max)$ and $\pi(\min) < \tilde{\pi}(\min)$. Then the comparison with case 1 will imply $\Delta > 0$.

We say that Y_n goes right (left resp.), if in the notations of this section $\eta_* = +1$ (-1 resp.). The analogous terminology is used for \tilde{Y}_n .

The rules of the coupling are quite standard and are as follows. If given $Y_{n-1} = \eta$ the probability to go right is x , where $x = p, q$, or $\frac{1}{2}$ resp. (η is local maximum, or minimum, or neither resp.), then Y_n goes right if $U_n \leq x$ and goes left otherwise. Similarly, \tilde{Y}_n goes to the right from the state $\tilde{\eta}$ if and only if $U_n \leq \tilde{x}$ where $\tilde{x} = \frac{1}{2}, \tilde{q}$ or \tilde{p} depending on the state $\tilde{\eta}$.

For $\eta, \tilde{\eta} \in \mathfrak{E}^{(L)}$ we write $\eta \succeq \tilde{\eta}$ whenever $\eta^i \geq (\tilde{\eta})^i$ for all $i = 1, 2, \dots, L$. Start with $Y_0 = \tilde{Y}_0$. Then it is easy to see by induction that $Y_n \succeq \tilde{Y}_n$ for all n . Indeed, $Y_0 \succeq \tilde{Y}_0$. Next, if $Y_m \succeq \tilde{Y}_m$ then

(a) if Y_m is a local minimum then \tilde{Y}_m is also a local minimum and hence $\tilde{\eta} \leq \eta$ with the strict inequality whenever $\tilde{q} < U_m \leq q$;

(b) if Y_m is a local maximum then $\tilde{\eta} \leq \eta$ since $p \geq \max(\tilde{p}, \frac{1}{2}, \tilde{q})$

(c) if Y_m is neither of the above then \tilde{Y}_n cannot be local maximum, and again $\tilde{\eta} \leq \eta$ since $\frac{1}{2} \geq \max(\frac{1}{2}, \tilde{q})$.

In all three cases above we have $Y_{m+1} \succeq \tilde{Y}_{m+1}$.

Finally, “ Y_n is a local minimum” implies “ \tilde{Y}_n is a local minimum” and also “ \tilde{Y}_n is a local maximum” implies “ Y_n is in a local maximum”, whence $\pi(\max) \geq \tilde{\pi}(\max)$ and $\pi(\min) \leq \tilde{\pi}(\min)$. Moreover, the event $\{\tilde{q} < U_n \leq q\}$ has a positive probability, consequently a positive fraction of times \tilde{Y}_n will achieve new local minimum right after it is a local minimum, while Y_n will go the right. Hence, $\tilde{\pi}(\min) > \pi(\min)$.

Case 2c: $p + q > 1, p \leq \frac{1}{2} \leq q$. Here we will have to construct a series of couplings, as the argument of Case 2b unfortunately cannot be applied directly.

We will also work directly with X_n rather than with Y_n ; clearly, $\pi(\max)$ and $\pi(\min)$ are the values determined also by the process X_n .

First, we construct the second process $\{\tilde{X}_n\}$ similar to X_n with $\tilde{q} = q$ but with $\tilde{p} = 1 - q < p$. Observe that the process \tilde{X}_n is symmetric and define the corresponding Markov chain \tilde{Y}_n for \tilde{X}_n in the same way Y_n was defined. Start with $X_0 = 0 = \tilde{X}_0$. Draw a *Uniform*(0, 1) random variable and go to the right if this variable is less than 1/2 and left otherwise. At the k -th stage, draw an independent *Uniform*(0, 1) random variable, and move to the right for the asymmetric chain, $\{X_n, Y_n\}$

$$\left\{ \begin{array}{ll} \text{if } U_k < p & \text{whenever } X_{k-1} \text{ is local max;} \\ \text{if } U_k < q & \text{whenever } X_{k-1} \text{ is local min;} \\ \text{if } U_k < \frac{1}{2} & \text{whenever } X_{k-1} \text{ is neither.} \end{array} \right.$$

Similarly, move to the right for the symmetric chain, $\{\tilde{X}_n, \tilde{Y}_n\}$,

$$\begin{cases} \text{if } U_k < \tilde{p} & \text{whenever } \tilde{X}_{k-1} \text{ is local max;} \\ \text{if } U_k < \tilde{q} = q & \text{whenever } \tilde{X}_{k-1} \text{ is local min;} \\ \text{if } U_k < \frac{1}{2} & \text{whenever } \tilde{X}_{k-1} \text{ is neither.} \end{cases}$$

Observe that both processes follow the same path till they hit a local maximum, say, at the $(k-1)$ -st step. Then the symmetric process has the smaller probability \tilde{p} to move to the right (i.e., another local maxima) than the original asymmetric process. In fact, if they make different moves, then the asymmetric process satisfies $X_k = \tilde{X}_k + 2$.

Our second step is to show by induction that $X_k \geq \tilde{X}_k$, in fact, for all k . Assume that $\tilde{X}_k \leq X_k$, for all $k \leq m$, and if for some k , $\tilde{X}_k < X_k$, then $\tilde{X}_k + 2 \leq X_k$. We now show that asymmetric process goes ahead of the symmetric process.

In the table below we write all the possibilities for the two processes at the m -th step, and then in each cell we write the pair of probabilities to move to the right. The first number in the brackets is the probability for X_n and the second number is the one for the symmetric process \tilde{X}_n .

$X_m \setminus \tilde{X}_m$	local max.	neither	local min.
local max.	(p, \tilde{p})	$(p, \frac{1}{2})$	(p, q)
neither	$(\frac{1}{2}, \tilde{p})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, q)$
local min.	(q, \tilde{p})	$(q, \frac{1}{2})$	(q, q)

In all the cells of lower triangular positions including diagonals (i.e., cells with coordinates $(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)$) the probability of moving to the right for the X_n is bigger than or equal to the probability of moving to the right for the \tilde{X}_n , hence $X_{m+1} \geq \tilde{X}_{m+1}$ for these cells.

Now for cells $(1, 2)$ or $(1, 3)$, if X_m is at the local maximum and $\tilde{X}_m = X_m$, then \tilde{X}_m must also be at the local maximum (since $\tilde{X}_k \leq X_k, \forall k \leq m$ by the assumption of induction), which contradicts the fact that \tilde{X}_m is not at local maximum. Therefore, $\tilde{X}_m + 2 \leq X_m$. Hence at the $(m+1)$ -st step if the processes do move not in the same direction, then they would be at most equal, i.e., $\tilde{X}_{m+1} \leq X_{m+1}$.

Finally, for cell $(2, 3)$, if \tilde{X}_m is at local minimum and $X_m = \tilde{X}_m$, then X_m must also be at the local minimum (since $X_k \geq \tilde{X}_k, \forall k \leq m$), which contradicts the fact that X_m is at neither local max nor local min. Thus again $\tilde{X}_m + 2 \leq X_m$ and if they make opposite moves at the $(m+1)$ -st step they would be at most equal: $\tilde{X}_{m+1} \leq X_{m+1}$ as before.

Hence we conclude the proof by induction that $X_k \geq \tilde{X}_k$ for all k and some times $\tilde{X}_k + 2 \leq X_k$.

The third step consists in using regeneration arguments. Let τ_1 be the first time k when $Y_k = \tilde{Y}_k = (+1, +1, \dots, +1)$, which is obviously finite as it is stochastically bounded by a geometric random variable with parameter $\max(p, q, \tilde{p}, \tilde{q}, 0.5)$. At time τ_1 construct a copy $\tilde{X}_k^{(1)}$ of the process $\tilde{X}_k^{(0)} := \tilde{X}_k$, such that $\tilde{Y}_{\tau_1}^{(1)} = Y_{\tau_1}$. By the arguments of the second step, $\tilde{X}_k \geq \tilde{X}_k^{(1)}$ for all $k \geq \tau_1$. Now let

$$\tau_2 = \min\{k > \tau_1 : Y_k = \tilde{Y}_k^{(1)} = (+1, +1, \dots, +1)\}$$

which is finite by the argument above, and construct another copy $\tilde{X}_k^{(2)}$ of the process \tilde{X}_k , now such that $\tilde{Y}_{\tau_2}^{(1)} = Y_{\tau_2}$. Repeating this procedure indefinitely we construct the sequence of stopping times $\tau_1, \tau_2, \tau_3, \dots$ and the sequence of processes $\tilde{X}_k^{(1)}, \tilde{X}_k^{(2)}, \tilde{X}_k^{(3)}, \dots$. Moreover, $\tau_{n+1} - \tau_n$, $n \geq 1$ are i.i.d. with finite expectation, say $\bar{\tau}$. Also, $\tilde{X}_k \geq \tilde{X}_k^{(n)}$ for all $k \geq \tau_n$, and because of the arguments of step two, $\tilde{X}_{\tau_n}^{(n-1)} - \tilde{X}_{\tau_n}^{(n)} \geq 2$ with a positive probability; moreover, these increments are nonnegative and independent for different n .

Observing that

$$\sum_{m=1}^n \left[\tilde{X}_{\tau_m}^{(m)} - \tilde{X}_{\tau_m}^{(m-1)} \right] = X_{\tau_n} - \tilde{X}_{\tau_n}$$

and using the strong law we obtain

$$\begin{aligned} \frac{\tau_n}{n} &\rightarrow \bar{\tau} && \text{a.s.}, \\ \frac{X_{\tau_n} - \tilde{X}_{\tau_n}}{n} &\rightarrow \alpha > 0 && \text{a.s.} \end{aligned}$$

for some constant α . This, in turn, implies that

$$\frac{X_k - \tilde{X}_k}{k} \rightarrow \alpha/\bar{\tau} > 0$$

after applying the renewal law of large numbers (Theorem 1.7.3 from [8]) for the sequence $\{\tau_k\}$.

Finally, in our fourth step, we use Lemma 1 and our result for the symmetric Case 1 applicable to \tilde{X}_k , to conclude that $\Delta > \tilde{\Delta} = 0$. Note that this step is only required for the proof of Lemma 2 and is not needed for the following Theorem 2, as the transience of X_n follows immediately from step 3 above (and, of course, Case 1).

Case 3a: $p + q < 1$, $p, q \leq \frac{1}{2}$. In this case, $2p - 1 \leq 0$ and $2q - 1 \leq 0$ and at least one of the inequalities is strict. Consequently, $\Delta = (2p - 1)\pi(\max) + (2q - 1)\pi(\min) < 0$.

Case 3b: $p + q < 1$, $p \leq \frac{1}{2} \leq q$. The proof is exactly similar to that of Case 2c with role of p replaced by $1 - q$ and q by $1 - p$, yielding $\Delta = (2p - 1)\pi(\max) + (2q - 1)\pi(\min) < 0$.

Case 3c: $p + q < 1$, $p \geq \frac{1}{2} \geq q$. In this case, the proof is exactly similar to that of Case 2b with role of p replaced by $1 - q$ and q by $1 - p$, whence $\Delta = (2p - 1)\pi(\max) + (2q - 1)\pi(\min) < 0$.

■

We now give the criteria for recurrence / transience of the finite memory chain.

Theorem 2 *For any $L \geq 1$, the finite memory chain is recurrent if and only if $p + q = 1$.*

We must note that even when the chain is recurrent but $p \neq 1/2$, the behaviour of the walk is different from that of a simple random walk. In particular, the variability increases for higher values of $p(= 1 - q)$. See Figure 1.

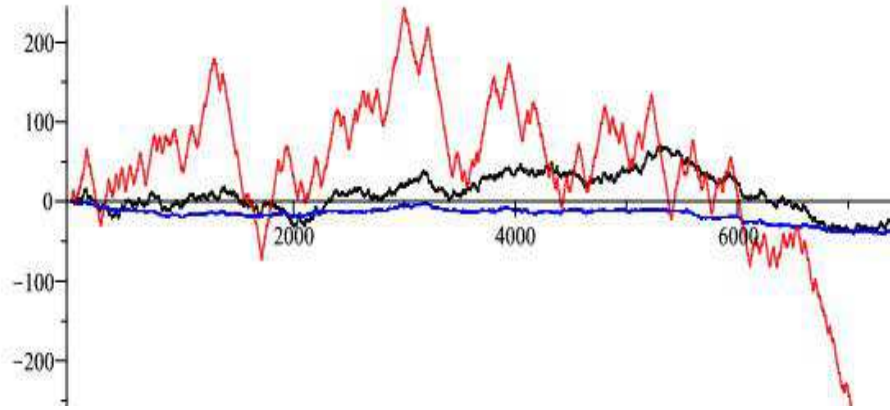


Figure 1: Trajectories of recurrent walks: red: $p = 1 - q = 0.9$, black: $p = q = 1/2$, blue: $p = 1 - q = 0.1$.

Proof of Theorem 2: If $p + q \neq 1$ then by Lemma 2 $\Delta \neq 0$ and hence by Lemma 1 $|X_n| \rightarrow \infty$ a.s.

If $p + q = 1$ then the chain (X_n, Y_n) is symmetric with respect to the change $-1 \leftrightarrow +1$. Let $T_{+\infty} = \{X_n \rightarrow +\infty\}$ and $T_{-\infty} = \{X_n \rightarrow -\infty\}$. By symmetry, $\mathbb{P}(T_{+\infty}) = \mathbb{P}(T_{-\infty})$. On the other hand, $\{X_n \rightarrow +\infty\}$ is a tail event, since, for example, there are infinitely many regeneration times when $X_i = X_{i-1} + 1$ for $i = n, n - 1, n - 2, \dots, n - L + 1$. Therefore, by Kolmogorov's zero-one law $\mathbb{P}(T_{+\infty}) \in \{0, 1\}$. Hence $\mathbb{P}(T_{+\infty}) = \mathbb{P}(T_{-\infty}) = 0$ and $\mathbb{P}(|X_n| \not\rightarrow \infty) = 1$. Consequently, there is a $k \in \mathbb{Z}$ such that $X_n = k$ for infinitely many n . And every time the walk hits k , the probability it will reach 0 in k steps is at least $[\min(p, 1 - p, q, 1 - q, \frac{1}{2})]^{|k|} > 0$. This implies the recurrence of X_n . ■

Theorem 3 *For any $L \geq 1$, the finite memory chain satisfies the central limit theorem, that is*

$$\frac{X_n - n\Delta_L}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \sigma^2)$$

where $\sigma^2 = \text{Var}_\pi(f(Y_1)) + 2 \lim_{\{m \rightarrow \infty\}} \sum_{k=1}^{m+1} \text{Cov}_\pi(f(Y_1), f(Y_{k+1}))$.

Proof: For f defined as in Lemma 1, use Functional CLT on positive recurrent Markov chains $\{Y\}$ (see, Theorem 10.2, p.150 of [4]) to get

$$\frac{X_n - n\Delta_L}{\sqrt{n}} = \frac{1}{\sqrt{n}} \left[\sum_{m=1}^n (f(Y_m) - \mathbb{E}_\pi f(Y_1)) + X_0 \right] \implies \mathcal{N}(0, \sigma^2)$$

where $\sigma^2 = \text{Var}_\pi(f(Y_1)) + 2 \lim_{\{m \rightarrow \infty\}} \sum_{k=1}^{m+1} \text{Cov}_\pi(f(Y_1), f(Y_{k+1}))$ whenever the limit exists and is finite, which holds for a finite state-space Markov Chain $\{Y\}$. ■

4 Conjectures and open problems

Here we list a few open problems and conjectures at which we have arrived by mostly looking at simulations of the process.

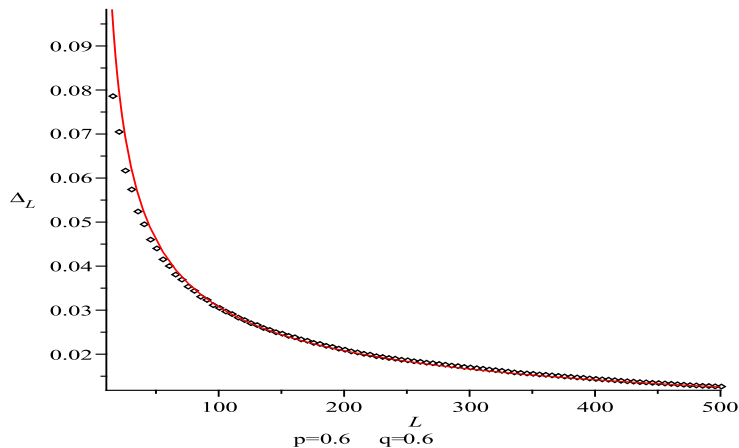


Figure 2: The speed Δ_L as a function of L vs. $const \cdot (2L \log \log L)^{-1/2}$.

In the transient case, when $p + q \neq 1$, the numerical simulations suggest that for fixed p and q we have

$$\Delta_L \propto \frac{1}{\sqrt{2L \log \log L}}$$

(see Figure 2). We believe that this order of magnitude corresponds to the fact that the range of the walk within the last L steps is of order \sqrt{L} and hence the frequency at which it visits the local maxima and minima, where it gets “a push” is something like $L^{-1/2}$ but unfortunately we do not have proof of this fact. The intuition behind this is that for a simple random walk ($p = q = 1/2$) the probability to be at the maxima is asymptotically $1/\sqrt{\pi L}$; this follows from Theorem 1.a in Chapter XII.8 and Theorem 1 in Chapter XVIII.5 in [9].

Also, we conjecture that Δ_L depends not only on “drift” $p + q - 1$, but in a complicated way on both p and q , see Figure 3 where in both cases the walk is transient.

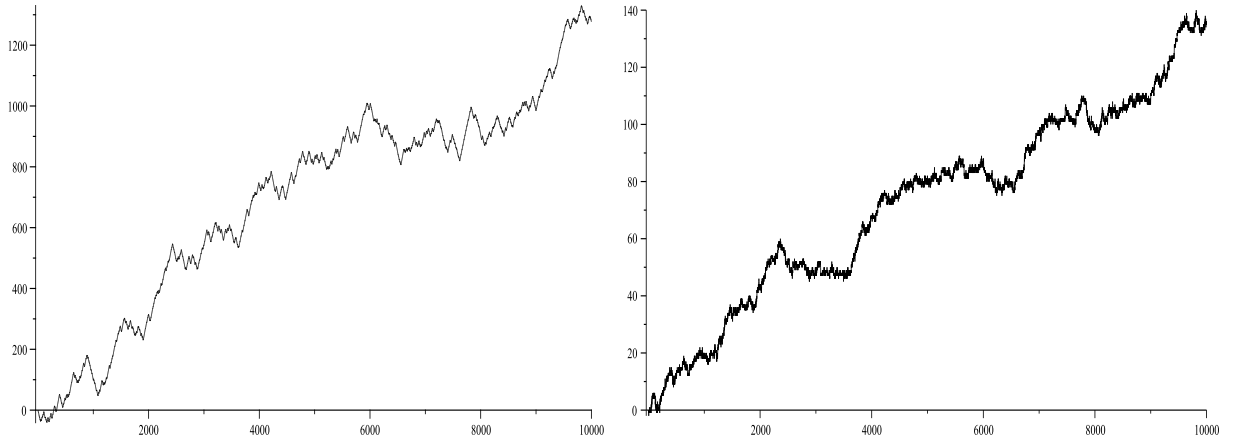


Figure 3: Left: $p = 0.95$, $q = 0.15$. Right: $q = 0.95$, $p = 0.15$.

Recall that in general we have $\Delta_L = (2p - 1)\pi(\max) + (2q - 1)\pi(\min)$, so estimating $\pi(\max)$ and $\pi(\min)$ is crucial in order to get the speed of the walk. We have another conjecture justified numerically: if $q = 1/2$ (so the walk is not perturbed at the minima) then

$$\pi_{\max} = \frac{1}{1 + a_L(1 - p)}, \quad a_L \sim L^{1/2}.$$

Again, we do not have a rigorous argument for this, and it would be hence nice to obtain a rigorous proof of this asymptotic dependence.

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