LIMITING SPECTRAL DISTRIBUTION OF REVERSE CIRCULANT MATRIX WITH DEPENDENT ENTRIES

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Abstract. In this article, we derive the limiting spectral distribution of the reverse circulant matrix when the input sequence is a stationary infinite order two sided moving average process.

Keywords: Large dimensional random matrix, eigenvalues, reverse circulant matrix, empirical spectral distribution, limiting spectral distribution, moving average process, convergence in distribution, convergence in probability, normal approximation.

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1. Introduction and Main result

Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ are all the real eigenvalues of a real symmetric square matrix $A_n$ of order $n$. Then the empirical spectral distribution function (ESDF) of $A_n$ is defined as

\[ F_n(x) = n^{-1} \sum_{i=1}^{n} I\{\lambda_i \leq x\}. \]

Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of square matrices with the corresponding ESDF \( \{F_n\}_{n=1}^{\infty} \). The Limiting Spectral Distribution (or measure) (LSD) of the sequence is defined as the weak limit of the sequence \( \{F_n\}_{n=1}^{\infty} \), if it exists.

If \( \{A_n\} \) are random, the limit is understood to be in some probabilistic sense, such as “almost surely” or “in probability”. Suppose elements of \( \{A_n\} \) are defined on some probability space \((\Omega, \mathcal{F}, P)\), that is \( \{A_n\} \) are random. Let $F$ be a nonrandom distribution function. We say the ESD of $A_n$ converges to the limiting spectral distribution (LSD) $F$ in $L_2$ if

\[ \int_{\omega} (F_n(x) - F(x))^2 dP(\omega) \to 0 \text{ as } n \to \infty \]

and converges in probability to $F$ if for every $\epsilon > 0$

\[ P(|F_n(x) - F(x)| > \epsilon) \to 0 \text{ as } n \to \infty. \]

For detailed information on limiting spectral distributions of large dimensional random matrices see [Bai(1999)] and also [Bose and Sen (2008)].
In this article we focus on obtaining the LSD of the reverse circulant matrix \( (RC_n) \) is given by

\[
RC_n = \frac{1}{\sqrt{n}} \begin{bmatrix}
x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\
x_1 & x_2 & x_3 & \cdots & x_0 & x_n \\
x_2 & x_3 & x_4 & \cdots & x_1 & x_0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n-1} & x_0 & x_1 & \cdots & x_{n-3} & x_{n-2}
\end{bmatrix}.
\]

So, the \((i, j)th\) element of the matrix is \(x_{(i+j-2)\text{mod} \ n}\). It is not hard to calculate the eigenvalues of \(RC_n\), for example see [Bose and Mitra(2002)]. The eigenvalues are given by:

\[
\begin{cases}
\lambda_0 &= n^{-1/2} \sum_{t=0}^{n-1} x_t \\
\lambda_{n/2} &= n^{-1/2} \sum_{t=0}^{n-1} (-1)^t x_t, \text{ if } n \text{ is even} \\
\lambda_k &= -\lambda_{n-k} = \sqrt{I_n(\omega_k)}, 1 \leq k \leq \lfloor n/2 \rfloor.
\end{cases}
\]

where,

\[
\omega_k = \frac{2\pi k}{n}, \quad I_n(\omega_k) = \frac{1}{n} \sum_{t=0}^{n-1} |x_t e^{-i\omega_k}|^2.
\]

Note that \([x]\) is the largest integer less than or equal to \(x\). The existence of the LSD of \(RC_n\) is given by the following theorem of [Bose and Mitra(2002)].

**Theorem 1.1.** Let \(\{x_i\}\) be a sequence of independent random variables with mean 0 and variance 1 and \(\sup_i E |x_i|^3 < \infty\). Then the ESD of \(RC_n\) converges in \(L_2\) to \(F\) with density \(f\) given by

\[
f(x) = |x| \exp(-x^2), \quad -\infty < x < \infty.
\]

We investigate the existence of LSD of this matrix under a dependent situation. Let \(\{x_n; n \geq 0\}\) be a two sided moving average process,

\[
x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}
\]

where \(\{a_n; n \in \mathbb{Z}\} \in l_1\), that is \(\sum_n |a_n| < \infty\), are nonrandom and \(\{\epsilon_i; i \in \mathbb{Z}\}\) are iid random variables with mean zero and variance one. We show that the LSD of \(RC_n\) continues to exist in this dependent situation. Define \(\gamma_h = Cov(x_{t+h}, x_t)\). Then it is easy to see that \(\sum_{j \in \mathbb{Z}} |\gamma_j| < \infty\) and the spectral density function of \(\{x_n\}\) is given by

\[
f(\omega) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \gamma_k \exp(ik\omega) = \frac{1}{2\pi} \left[ \gamma_0 + 2 \sum_{k \geq 1} \gamma_k \cos(k\omega) \right] \text{ for } \omega \in [0, 2\pi].
\]

Let \(f^* = \inf_{\omega \in [0, 2\pi]} f(\omega)\). For \(k = 1, 2, \cdots, \lfloor n/2 \rfloor\), define

\[
\omega_k = \frac{2\pi k}{n}, \quad \xi_{2k-1} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \epsilon_t \cos(\omega_k t), \quad \xi_{2k} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \epsilon_t \sin(\omega_k t),
\]
Then for satisfying the distribution of the two eigenvalues distribution. So, it is enough to concentrate on the case from the structure of the eigenvalues, the LSD, if it exists, is going to that of a symmetric LSD, if it exists, is going to that of a symmetric

\[
I_n(\omega_k) = \frac{1}{n} \left| \sum_{t=0}^{n-1} x_t e^{-i t \omega_k} \right|^2 = \frac{1}{n} \left[ \left( \sum_{t=0}^{n-1} x_t \cos(\omega_k t) \right)^2 + \left( \sum_{t=0}^{n-1} x_t \sin(\omega_k t) \right)^2 \right].
\]

**Theorem 1.2.** Suppose \( \{\epsilon_t\} \) are iid with \( E|\epsilon_t|^{(2+\delta)} < \infty \). Then the ESD of \( RC_n \) converges in \( L_2 \) to the LSD \( H(x) = \begin{cases} 
1 - \int_0^\pi \frac{1}{2\pi} e^{-\frac{x^2}{2\pi f(\omega)}} d\omega & \text{if } x > 0 \\
\int_0^\pi \frac{1}{2\pi} e^{-\frac{x^2}{2\pi f(\omega)}} d\omega & \text{if } x \leq 0.
\end{cases} \)

It may be noted that integrand is zero whenever \( f(\omega) = 0 \).

Proof of Theorem mainly depends on following two lemmas. The proof of Lemma 1.3 is given in [Fan and Yao(2003)] (Theorem 2.14(ii), page 63). The proof of Lemma 1.4 follows easily from [Bhattacharya and Ranga Rao(1976)] (Corollary 18.3, page 184).

**Lemma 1.3.** Let \( x_t = \sum_{j=-\infty}^{\infty} a_t \epsilon_{t-j} \) for \( t \geq 0 \), where \( \{\epsilon_t\} \sim IID(0,1) \) and \( \sum_{j=-\infty}^{\infty} |a_j| < \infty \). Then for \( k = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor \),

\[
I_n(\omega_k) = 2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) + R_n(\omega_k)
\]

and \( \max_{1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor} E|R_n(\omega_k)| \to 0 \) as \( n \to \infty \).

**Lemma 1.4.** Let \( X_1, \ldots, X_k \) be independent random vectors with values in \( \mathbb{R}^d \), having zero means and an average positive-definite covariance matrix \( V_k = k^{-1} \sum_{j=1}^k \text{Cov} X_j \). Let \( G_k \) denote the distribution of \( k^{-1/2} T_k (X_1 + \ldots + X_k) \), where \( T_k \) is the symmetric, positive-definite matrix satisfying \( T_k^2 = V_k^{-1}, n \geq 1 \). If for some \( \delta > 0 \), \( E \| X_j \|^{(2+\delta)} < \infty \), then

\[
\sup_{C \in \mathcal{C}} |G_k(C) - \Phi_{0,1}(C)| \leq \ c k^{-\delta/2} \left[ k^{-1} \sum_{j=1}^k E \| T_k X_j \|^{(2+\delta)} \right]
\]

\[
\leq \ c k^{-\delta/2} (\lambda_{\min}(V_k))^{-(2+\delta)} \left[ k^{-1} \sum_{j=1}^k E \| X_j \|^{(2+\delta)} \right]
\]

where \( \Phi_{0,1} \) is the normal probability function with mean zero and identity covariance matrix, \( \mathcal{C} \), the class of all Borel-measurable convex subsets of \( \mathbb{R}^d \) and \( c \) is a constant, depending only on \( d \).

**Proof of Theorem 1.2:** To prove the theorem it suffices to show that for each \( x \),

\[
E(F_n(x)) \to H(x) \quad \text{and} \quad V(F_n(x)) \to 0.
\]

From the structure of the eigenvalues, the LSD, if it exists, is going to that of a symmetric distribution. So, it is enough to concentrate on the case \( x > 0 \). Also note that we may ignore the two eigenvalues \( \lambda_0 \) and \( \lambda_{n/2} \) since they contribute \( 2/n \) to the ESD \( F_n \).
Hence for $x > 0$,

$$E(F_n(x)) \sim 1/2 + n^{-1} \sum_{k=1}^{\lfloor n/2 \rfloor} P(I_n(\omega_k) \leq x^2).$$

From Lemma 1.3, it is intuitively clear that for large $n, I_n(\omega_k) \sim 2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2).$ So first we show that for large $n$

$$\frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} P(I_n(\omega_k) \leq x^2) \sim \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \leq x^2).$$

Let $L_n(\omega_k) = 2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2)$ for $1 \leq k \leq \lfloor n/2 \rfloor$. Then

$$\left| \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} P(I_n(\omega_k) \leq x^2) - \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \leq x^2) \right|$$

$$\leq \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} |P(L_n(\omega_k) + R_n(\omega_k) \leq x^2) - P(L_n(\omega_k) \leq x^2)|$$

$$\leq \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} P(|R_n(\omega_k)| \geq \epsilon) + \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} |P(L_n(\omega_k) \leq x^2) - P(L_n(\omega_k) \leq x^2 \pm \epsilon)|$$

$$\leq T_1 + T_2 + T_3 + T_4,$$

where

$$T_1 = \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} P(|R_n(\omega_k)| \geq \epsilon), \quad T_2 = \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} |P(L_n(\omega_k) \leq x^2) - \Phi_{0,1}(A_{kn})|,$$

$$T_3 = \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} |P(L_n(\omega_k) \leq x^2 \pm \epsilon) - \Phi_{0,1}(A_{kn}^\epsilon)|, \quad T_4 = \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} |\Phi_{0,1}(A_{kn}) - \Phi_{0,1}(A_{kn}^\epsilon)|,$$

$$A_{kn} = \{(r_1, r_2) : \frac{r_1^2}{2} + \frac{r_2^2}{2} \leq \frac{x^2}{2\pi f(\omega_k)}\}, \quad A_{kn}^\epsilon = \{(r_1, r_2) : \frac{r_1^2}{2} + \frac{r_2^2}{2} \leq \frac{x^2 \pm \epsilon}{2\pi f(\omega_k)}\}.$$

For convenience we assume that $f^* > 0$. If there exists $\omega$ such that $f(\omega) = 0$, then the proof given below can be easily modified.

Now as $n \to \infty$,

$$T_1 \leq \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} \epsilon^{-1} E|R_n(\omega_k)| \leq \frac{1}{\epsilon} \max_{1 \leq k \leq \lfloor n/2 \rfloor} E|R_n(\omega_k)| \to 0.$$

$$T_3 \leq \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} \epsilon e^{-\frac{(x^2 - \epsilon)}{2\pi f(\omega_k)}} \leq C \epsilon,$$

where $C$ is a constant and right side can be make arbitrarily small by choosing $\epsilon$ small enough.

To show $T_2, T_3 \to 0$ define for $k = 1, 2, \cdots, \lfloor n/2 \rfloor$ and $l = 0, 1, 2, \cdots, n - 1$, ...
\[ X_{l,k} = (\sqrt{2}e_l \cos(l\omega_k), \ \sqrt{2}e_l \sin(l\omega_k))'. \]

Note that

\begin{equation}
E(X_{l,k}) = 0 \ \forall \ l, k, n. \tag{1.1}
\end{equation}

\begin{equation}
n^{-1} \sum_{l=0}^{n-1} \text{Cov}(X_{l,k}) = I \ \forall \ k, n. \tag{1.2}
\end{equation}

Note that

\[ \{2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \leq x^2\} = \{n^{-1/2} \sum_{l=0}^{n-1} X_{l,k} \in A_{kn}\}. \]

Since \( A_{kn} \) is a convex set in \( \mathbb{R}^2 \) and since \( \{X_{l,k}, l = 0, 1, \ldots (n - 1)\} \) satisfies (1.1) and (1.2), we can apply Lemma 1.4 to get

\[ |P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \leq x^2) - \Phi_{0,l}(A_{kn})| \leq cn^{-\delta/2}[n^{-1} \sum_{l=0}^{n-1} E \|X_{l,k}\|^{(2+\delta)}] \]

Note

\[ \sup_{1 \leq k \leq \left[\frac{n-1}{2}\right]} |n^{-1} \sum_{l=0}^{n-1} E \|X_{l,k}\|^{(2+\delta)}| \leq M < \infty. \]

\[ I_1 = \frac{1}{n} \sum_{k=1}^{\left[\frac{n-1}{2}\right]} |P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \leq x^2) - \Phi_{0,l}(A_{kn})| \leq cMn^{-\delta/2} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Hence \( T_2 \rightarrow 0 \) and similarly \( T_3 \rightarrow 0 \). Therefore

\[ E(F_n(x)) \sim 1/2 + n^{-1} \sum_{k=1}^{\left[\frac{n-1}{2}\right]} P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \leq x^2), \]

and also

\[ \left|\frac{1}{n} \sum_{k=1}^{\left[\frac{n-1}{2}\right]} P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \leq x^2) - \Phi_{0,l}(A_{kn})\right| \rightarrow 0 \text{ as } n \rightarrow \infty. \]
Now
\[
\frac{1}{n} \sum_{k=1}^{[n/2]} \Phi_{0,k}(A_{kn}) = \frac{1}{n} \sum_{k=1}^{[n/2]} \left( 1 - e^{-\frac{x^2}{2\pi f(\omega_k)}} \right)
\]
\[
= \frac{1}{n} \sum_{k=1}^{[n/2]} \left( 1 - e^{-\frac{x^2}{\gamma_0 + 2 \sum_{h=1}^{\infty} \gamma_h \cos 2\pi n h}} \right)
\]
\[
\rightarrow \frac{1}{2} - \int \frac{1}{2\pi} e^{-\frac{x^2}{\gamma_0 + 2 \sum_{h=1}^{\infty} \gamma_h \cos 2\pi n h}} dt
\]
\[
= \frac{1}{2} - \int \frac{\pi}{2\pi} e^{-\frac{x^2}{\gamma_0 + 2 \sum_{h=1}^{\infty} \gamma_h \cos 2\pi n h}} d\omega = G(x), \text{ say.}
\]
Hence for \( x \geq 0 \),
\[
E(F_n(x)) \rightarrow 1 - \int_0^{\pi} \frac{1}{2\pi} e^{-\frac{x^2}{\gamma_0 + 2 \sum_{h=1}^{\infty} \gamma_h \cos 2\pi n h}} d\omega = H(x).
\]
Now, to show \( V(F_n(x)) \rightarrow 0 \), it is enough to show that
\[(1.3) \quad \frac{1}{n^2} \sum_{k \neq k', k, k' = 1}^{[n/2]} \text{Cov}(J_k, J_{k'}) \to 0. \]
where for \( 1 \leq k \leq [n/2] \), \( J_k \) is the indicator that \( \{ I_n(\omega_k) \leq x^2 \} \).
\[
\frac{1}{n^2} \sum_{k \neq k', k, k' = 1}^{[n/2]} \text{Cov}(J_k, J_{k'}) = \frac{1}{n^2} \sum_{k \neq k', k, k' = 1}^{[n/2]} \left[ E(J_k, J_{k'}) - E(J_k)E(J_{k'}) \right].
\]
Now as \( n \to \infty \),
\[
\frac{1}{n^2} \sum_{k \neq k', k, k' = 1}^{[n/2]} E(J_k)E(J_{k'}) = \left( \frac{1}{n} \sum_{k=1}^{[n/2]} E(J_k) \right)^2 - \frac{1}{n^2} \sum_{k=1}^{[n/2]} (E(J_k))^2 \to G(x)^2.
\]
So to show \( (1.3) \), it is enough to show as \( n \to \infty \),
\[
\frac{1}{n^2} \sum_{k \neq k', k, k' = 1}^{[n/2]} E(J_k, J_{k'}) \to G(x)^2.
\]
Along the lines of the proof used to show \( \frac{1}{n} \sum_{k=0}^{[n/2]} P(I_n(\omega_k) \leq x^2) \to G(x) \), one may now extend the vectors of two coordinates defined above to ones with four coordinates and proceed exactly as above to verify this. We omit the routine details. \( \square \)

**References**


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