HETEROSCEDASTIC WIGNER MATRICES

SAYAN BANERJEE
University of Washington, Seattle
sayan.banerjee20@gmail.com

ARUP BOSE *
Indian Statistical Institute, Kolkata
bosearu@gmail.com

SANCHAYAN SEN
Courant Institute
sanchayan1987@gmail.com

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Abstract

It is well known that the Wigner matrix with independent mean zero variance one entries that are either uniformly bounded or are identically distributed has a limiting spectral distribution which is the famous semicircular law.

A natural question is then what can be said if the entries have unequal variances. We investigate several such scenarios. In particular we show that if the entry sequence is $\{\sigma_{ij}(n)x_{ij} : i \leq j\}$, where $\sigma_{ij}(n)$'s are uniformly bounded and row averages of the $\{(\sigma_{ij}(n))^2\}$ converge uniformly to a common value, say c^2, then the LSD is semicircular with scale parameter c. Further, if $\sigma_{ij}(n) = \delta_i^{(n)}\delta_j^{(n)}$, where the $\delta_i^{(n)}$'s are uniformly bounded non-negative numbers such that $\lim_n \sum_{i=1}^{n}(\delta_i^{(n)})^k/n = c_k$ for all $k \geq 1$, then also, the LSD exists but need not be semicircular. We give a recursive expression for the moments of the limit and show that the limit is semicircular if and only if for some b, $c_k = b^k$ for all $k \geq 1$.

Keywords: Catalan words, eigenvalues, empirical spectral distribution (ESD), limiting spectral distribution (LSD), moment method, non-crossing partitions, semicircular law, Wigner matrix.

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1 INTRODUCTION

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a real symmetric matrix $A_{n\times n}$ then its *Empirical Spectral Distribution (ESD)* F^{A_n} is defined as

$$F^{A_n}(x) = n^{-1}\sum_{i=1}^{n}\mathbb{I}(\lambda_i \leq x). \tag{1.1}$$

The *Limiting Spectral Distribution (LSD)* of $\{A_n\}$ is the weak limit of $\{F^{A_n}\}$, if it exists, either almost surely or in probability.

The $n \times n$ Wigner matrix is the symmetric matrix $((x_{ij}))$ (with $x_{ij} = x_{ji}$ for $i > j$). We call $\{x_{ij}\}$ as the input sequence.

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It is a well known and celebrated fact that the LSD of a sequence of Wigner matrices having i.i.d. inputs with mean zero and finite variance is the semi-circular law given by

$$f(s) = \begin{cases} \frac{1}{2\pi}\sqrt{4-s^2} & \text{if } |s| \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

See Wigner (1958)[10], Bai (1999)[2] and Anderson, Guionnet and Zeitouni (2009)[1] for such results and their variations. In particular, if the entries have mean zero variance one and they are either identically distributed or uniformly bounded, then the semicircle limit holds. It is known that each limit moment is the total number of *non-crossing pair partitions* (see for example Anderson, Guionnet and Zeitouni (2009)[1]). Equivalently, it is also the total number of so called *Catalan words* (see Bose, Hazra and Saha (2010)[5]).

Banerjee and Bose (2010)[4] showed that a large class of homoscedastic matrices with very general structural assumptions (that includes the Wigner matrix) also yield the semicircular law as the LSD. The key to this is that even in such cases, the sole contribution to the limiting moments comes from the non-crossing pair partitions. Thus the semicircle law is robust with respect to the structure of the matrix to a certain extent. A natural question is to what extent there is robustness with respect to *heteroscedasticity* of the entries of the Wigner matrix. We investigate several such scenarios.

Consider the (symmetric) Wigner matrix ($\sigma_{ij}(n) = \sigma_{ij}(n)$ and $x_{ij} = x_{ji}$)

$$W_n = ((\sigma_{ij}(n)x_{ij})). \tag{1.3}$$

We show that if $\{x_{ij}\}$ satisfies Condition A (see below) and $\{\sigma_{ij}(n)\}$ satisfies (2.1) then the LSD is semicircular with scale parameter c. Further, if $\sigma_{ij}(n) = \delta_i^{(n)}\delta_j^{(n)}$, where $\{\delta_i^{(n)}\}$ satisfies (2.3), then also the LSD exists. However, it may not be semicircular. We provide a recursive expression for the limit moments and show that the limit is semicircular with scale parameter b^2, if and only if for some $b > 0$, $c_i = b^i$ for all i in (2.3). Preliminary version of these results appears in Banerjee (2010) [3] and Sen (2010) [9].

2 Main Results

In this section we state the main theorems, their corollaries and examples. The proofs of the theorems are give in the next section.

Condition A: $\{x_{ij} : i \le j\}$ are independent, mean zero variance one random variables such that either (a) for each k, $\sup\limits_{i\le j} E|x_{ij}|^k < \infty$, or (b) they are identically distributed.

Theorem 1 *Let $\{W_n\}$ be as in (1.3) where $\{x_{ij} : i \le j\}$ satisfies Condition A and for some $0 \le c, M < \infty$,*

$$\sup_{1\le i\le n} |\frac{1}{n}\sum_{j=1}^{n}(\sigma_{ij}(n))^2 - c^2| \to 0 \text{ as } n \to \infty \text{ and } \sup_{i,j,n}|\sigma_{ij}(n)| \le M. \tag{2.1}$$

Then LSD of $n^{-1/2}W_n$ exists almost surely and is cW, where W follows the semicircular law.

As an application of the above theorem, the following corollary which is a version of a result in Anderson and Zeitouni (2006) [11] is immediate.

Corollary 1.1 *Let f be a non-negative, symmetric, continuous function on $[0,1]^2$ such that $\int_0^1 f(x,y)dy \equiv 1$. Consider W_n as in (1.3) where $\sigma_{ij}(n) = [f(i/n, j/n)]^{1/2}$ and $\{x_{ij}\}$ $(i \ge j)$ satisfies Condition A. Then the LSD of $n^{-1/2}W_n$ is semicircular.*

Another interesting example is given in the following corollary:

Corollary 1.2 *Suppose $\{c_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ such that $|c_n| \leq M,\ \forall n$, (for some $M > 0$) and m_n is a sequence of integers and $n^{-1}|m_n N(\frac{n}{m_n}) - n| \to 0$. Further assume that as $n \to \infty$*

$$m_n^{-1} \sum_{j=1}^{m_n} c_j^2 \longrightarrow \alpha^2. \tag{2.2}$$

Let $\{Y_{i,j}\}_{1\leq i\leq j}$ be a triangular sequence of i.i.d. random variables with $\mathrm{E}(Y_{i,j}) = 0, \mathrm{E}(Y_{i,j}^2) = 1, \forall i, j$ and let $Y_{i,j} = Y_{j,i}$ for $i > j$. Define $X_{i,j}^{(n)} = c_{k+1} Y_{i,j}$ if $i + j = k(\text{ mod } m_n)$ $(0 \leq k \leq m_n - 1)$ and let

$$\overline{W}_n = \left(\left(X_{i,j}^{(n)}\right)\right)_{i,j=1,\ldots,n}$$

Then LSD of $n^{-1/2}\overline{W}_n$ exists almost surely and is cW, where W follows the semicircular law.

In particular, the above is true under any of the following conditions:

(i) $\frac{n}{m_n}$ is always an integer,

(ii) $\frac{m_n}{n} \to 0$,

(iii) $m_n = \left[\frac{n}{q}\right]$ for a fixed integer q.

We now consider the particular choice of $\sigma_{ij}(n) = \delta_i^{(n)}\delta_j^{(n)}$. This is same as considering the matrix $T_n W_n T_n$ where $T_n = Diag(\delta_1^{(n)}, \ldots, \delta_j^{(n)})$. Using Stieltjes transform method, Sen (2006) [8] proved that the LSD of $T_n W_n T_n$ exists when the input sequence is uniformly bounded, $T_n = Diag(t_1, \ldots, t_n)$ where $\{t_i\}$ is a uniformly bounded sequence and the LSD of T_n exists. The following theorem is an extension of that result and we give a moment method proof.

Theorem 2 *Let $\{W_n\}$ be as in (1.3) where $\{x_{ij} : i \leq j\}$ satisfies Condition A and $\sigma_{ij}(n) = \delta_i^{(n)}\delta_j^{(n)}$ with*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}(\delta_j^{(n)})^k = c_k \ \ (say) \ \ for\ each \ \ k \ \ and \ \ \sup_{i,n}|\delta_i^{(n)}| \leq M < \infty. \tag{2.3}$$

Then the LSD of $n^{-1/2}W_n$ exists, and its moments are given by

$$\beta_{2k} = \sum_{t=1}^{k} \tau_{2k-2t}\tau_{2t-2} \tag{2.4}$$

where $\{\tau_{2k}\}$ are recursively defined as

$$\tau_0 = c_1 \ \text{ and } \ \tau_{2k} = \sum_{p=1}^{k} \sum_{n_1+n_2+\cdots+n_p=k} c_{p+1} \prod_{i=1}^{p} \tau_{2n_i-2}.$$

Consequently, the generating functions

$$\phi_\tau(x) = \sum_{p=0}^{\infty} \tau_{2p}x^{2p}, \phi_\beta(x) = \sum_{p=0}^{\infty} \beta_{2p}x^{2p} \ and \ \phi_c(x) = \sum_{p=0}^{\infty} c_{p+1}x^p,$$

satisfy

$$\phi_\tau(x) = \phi_c(x^2\phi_\tau(x)) \text{ and } \phi_\beta(x) = 1 + [x\phi_\tau(x)]^2.$$

This LSD is semicircular if and only if $c_i = b^i$ for every i. In this case, the scale parameter is b^2.

The result continues to hold if $\delta_i^{(n)} = \delta_i$ is an independent sequence of uniformly bounded random variables which is independent of the input sequence.

Example: Let f be a bounded, non-negative, continuous function on $[0, 1]$. Set $\delta_i^{(n)} = f(\frac{i}{n})$. Then, using Riemann sums, $c_k = \int_0^1 f(t)^k dt$ in (2.3). Hence, in some interval around 0,

$$\phi_c(x) = \int_0^1 \frac{f(t)}{1 - xf(t)} dt.$$

Therefore, in some interval around 0, $\phi_\tau(x)$ satisfies the equation:

$$\phi_\tau(x) = \int_0^1 \frac{f(t)}{1 - x^2\phi_\tau(x)f(t)} dt.$$

For example, if we take $f(t) = t$, then $\phi_\tau(x)$ satisfies

$$1 + [x\phi_\tau(x)]^2 + \frac{\ln(1 - x^2\phi_\tau(x))}{x^2\phi_\tau(x)} = 0.$$

This equation can be solved numerically to any level of precision to find $\phi_\tau(x)$ and hence $\phi_\beta(x)$. The first few moments of the LSD using the recursion relations of the theorem are $\beta_2 = 1/4$, $\beta_4 = 1/6$, $\beta_6 = 7/48$.

Figure 1 shows the simulation of LSDs for three different choices of $\{\delta_i\}$.

3 PROOFS

By virtue of truncation methods (see for examples in Bose and Sen (2008)[6] or Bryc, Dembo and Jiang (2006)[7]), it is easy to see that under Condition A, we may assume that $\{x_{ij}\}$ is uniformly bounded. *Henceforth we assume this.*

To prove our theorems we shall use the method of moments: Suppose $\{A_n\}$ is a sequence of patterned random matrices, and let $\beta_h(A_n)$, for $h \geq 1$, denote the h-th moment of the ESD of A_n. Suppose there is a sequence of non-random $\{\beta_h\}_{h=1}^\infty$ such that,

(M1) For every $h \geq 1$, $\mathrm{E}[\beta_h(A_n)] \to \beta_h$

(M2) $\mathrm{E}[\beta_h(A_n) - \mathrm{E}(\beta_h(A_n))]^4 = \mathrm{O}(n^{-2})$.

Further assume that $\{\beta_h\}_{h=1}^\infty$ satisfies Carleman's condition

$$\sum_{h=1}^\infty \beta_{2h}^{-1/2h} = \infty. \tag{3.1}$$

Then the LSD is identified by $\{\beta_h\}_{h=1}^\infty$ and the convergence to LSD holds almost surely.

We shall verify (M1) and the Carleman's condition. The verification of (M2) is similar to the arguments given in Bose and Sen (2008) [5] and shall be omitted. We now mention some of the key concepts that we need from Bose and Sen (2008)[6]. Define the *link function* for the Wigner matrix as

$$L(i, j) = (\min(i, j), \max(i, j)).$$

Any function $\pi : \{0, 1, 2, \cdots, h\} \to \{1, 2, \cdots, n\}$ is a *circuit* if $\pi(0) = \pi(h)$. The *length* $l(\pi)$ of π is taken to be h. A circuit depends on h and n but we will suppress this dependence. Let

$$y_{ij}^{(n)} = \sigma_{ij}(n)x_{ij} \ (i \leq j) \ \text{ and } \ Y_\pi^{(n)} = \prod_{i=1}^{h} y_{L(\pi(i-1),\pi(i))}^{(n)}.$$

Any function of the trace may be written in terms of $Y_\pi^{(n)}$. For instance

$$\mathrm{E}[\beta_h(n^{-1/2}W_n)] = \frac{1}{n^{h/2+1}} \sum_{\pi} \mathrm{E}(Y_\pi^{(n)}). \tag{3.2}$$

Any value $L(\pi(i-1), \pi(i))$ is said to be an *L-value* of π. If π has at least two identical L-valuem then it is called *matched.* It is called *pair matched* if each L-value is repeated exactly two times. If π has at least one edge of order one then $E(Y_\pi^{(n)}) = 0$. Thus only matched circuits are relevant.

Two circuits π_1 and π_2 (of same length) are said to be *equivalent* if their L values agree at exactly the same pairs (i, j). That is, iff $\{L(\pi_1(i-1), \pi_1(i)) = L(\pi_1(j-1), \pi_1(j)) \Leftrightarrow L(\pi_2(i-1), \pi_2(i)) = L(\pi_2(j-1), \pi_2(j))\}$. This defines an equivalence relation between the circuits.

Equivalence classes may be identified with partitions of $\{1, 2, \cdots, h\}$: to any partition we associate a *word* w of length $l(w) = h$ of letters where the first occurrence of each letter is in alphabetical order. For example, if $h = 5$, then the partition $\{\{1, 3, 5\}, \{2, 4\}\}$ is represented by the word $ababa$.

Let $w[i]$ denote the i-th entry of w. The equivalence class corresponding to w is

$$\Pi(w) = \{\pi : w[i] = w[j] \Leftrightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\}.$$

The number of partition blocks corresponding to w will be denoted by $|w|$. If $\pi \in \Pi(w)$, then clearly, $\#\{L(\pi(i-1), \pi(i)) : 1 \leq i \leq h\} = |w|$.

The notions of matching, pair matching for π carries over to words. For instance, $ababa$ is matched (but not pair matched), $abcadbaa$ is non-matched and the corresponding partition is $\{\{1, 4, 7, 8\}, \{2, 6\}, \{3\}, \{5\}\}$. As pointed out, it will be enough to consider only matched words. The total number of pair matched words with $|w| = k$ equals $\frac{(2k)!}{2^k k!}$.

A pair matched word is said to be *Catalan* if it possesses a double letter xx and successive removal of such double letters reduces it to the empty word. This sets up a 1-1 correspondence with the *non-crossing pair partitions*. The number of Catalan words of length $2k$ equals $\frac{(2k)!}{k!(k+1)!}$.

Define for any (matched) word w,

$$\Pi^*(w) = \{\pi : w[i] = w[j] \Rightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\} \supseteqq \Pi(w).$$

Given π, any $\pi(i)$ (or i) is a *vertex*. It is *generating* if either $i = 0$ or $w[i]$ is the position of the *first* occurrence of a letter. For example, if $w = abbcab$ then $\pi(0), \pi(1), \pi(2), \pi(4)$ are generating vertices. By Property B, a circuit is completely determined, *up to a finitely many choices* by its generating vertices. The number of generating vertices in π is $|w| + 1$. Hence $\#\Pi^*(w) = \mathrm{O}(n^{|w|+1})$.

It is further known from Bose and Sen (2008)[6] that $\frac{1}{n^{1+k}}\#\Pi(w)$ and $\frac{1}{n^{1+k}}\#\Pi^*(w)$ have same limit behaviour.

In particular this limit equals one if w is Catalan and equals zero otherwise. Now, if $\sigma_{ij}(n) \equiv 1$, then it is also known that in the limit (3.2), only pair matched circuits contribute. Further, since for every pair matched circuit, $E(Y_\pi^{(n)}) = 1$, $\lim E\beta_h(n^{-1/2}W_n) = \#$ Catalan words which is the h^{th} moment of the semi circle law.

Now when $\sigma_{ij}(n)$'s are uniformly bounded, it also readily follows from these development that non-matched words do not contribute to the limit moments.

For each circuit π, define

$$\sigma_\pi(n) = \prod_{i=1}^{h} \sigma_{L(\pi(i-1),\pi(i))}(n).$$

Define, for each fixed matched word w of length $2k$ with $|w| = k$,

$$p(w) = \lim_n \frac{1}{n^{1+k}} \sum_{\pi\in\Pi^*(w)} \sigma_\pi(n) = \lim_n \frac{1}{n^{1+k}} \sum_{\pi\in\Pi(w)} \sigma_\pi(n) \tag{3.3}$$

whenever any of the two limit exists. As we have just discussed $p(w) = 0$ if w is not Catalan.

Therefore, if the limits $p(w)$ exist for each Catalan word w, then the limiting $(2k)$-th moment of $n^{-1/2}W_n$ is the finite sum

$$\lim \beta_{2k}(n^{-1/2}W_n) \equiv \beta_{2k} \equiv \sum_{w:l(w)=2k,\ w \text{ Catalan}} p(w)$$

and the odd moments, are zero. Further, $(M2)$ holds. We omit the proof of this. It is also easy to see that $p(w) \leq M^{2k}$ and hence $\{\beta_{2k}\}$ satisfies the Carleman's condition.

Thus it boils down to evaluating $p(w)$ for Catalan words w under different models for the $\{\sigma_{ij}(n)\}$.

3.1 Proof of Theorem 1

Take a Catalan word w of length $2k$. Let $w[j] = w[j+1]$. Let w' be the Catalan word obtained by deleting these two letters. Let

$$w_{i,n} = \frac{1}{n}\sum_{u=1}^{n} \sigma_{iu}(n)^2 - c^2.$$

Take any small enough $\epsilon > 0$, and choose N such that $\forall n \geq N$, $\sup_{1\leq i\leq n} |w_{i,n}| < \epsilon$. For each circuit $\pi \in \Pi^*(w)$ define

$$\pi' = (\pi(0), \pi(1), \ldots, \pi(j-1), \pi(j+2), \ldots, \pi(2k)).$$

As for the Wigner link function, $L(x,y) = L(y,z) \Rightarrow x = z$, therefore π' is a circuit and $\{\pi' : \pi \in \Pi^*(w)\} = \Pi^*(w')$, each π' being the restriction of exactly n circuits $\{\pi_1, \pi_2, \ldots, \pi_n\}$ in $\Pi^*(w)$ where $\pi_u = (\pi'(0), \pi'(1), \ldots, \pi'(j-1), u, \pi'(j-1), \pi'(j), \ldots, \pi'(2k))$. Now,

$$\begin{aligned}\left|\frac{1}{n^{k+1}} \sum_{\pi\in\Pi^*(w)} \sigma_\pi(n) - c^{2k}\right| \quad &\leq \quad \left|\frac{1}{n^{k+1}} \sum_{\pi\in\Pi^*(w)} \sigma_\pi(n) - \frac{c^2}{n^k} \sum_{\pi':\pi\in\Pi^*(w)} \sigma_{\pi'}(n)\right| \\ &\quad + c^2 \left|\frac{1}{n^k} \sum_{\pi':\pi\in\Pi^*(w)} \sigma_{\pi'}(n) - c^{2k-2}\right|.\end{aligned}$$

Also,

$$\frac{1}{n^{k+1}} \sum_{\pi\in\Pi^*(w)} \prod_{i=0}^{2k-1} \sigma_{\pi(i)\pi(i+1)}(n) \quad = \quad \sum_{\pi'\in\Pi^*(w')} \frac{\sigma_{\pi'(0)\pi'(1)}(n)}{n^{k+1}} \cdots \left(\sum_{u=1}^{n} \sigma_{\pi'(j-1)u}(n)\sigma_{u\pi'(j-1)}(n)\right)$$

$$\cdots \sigma_{\pi'(2k-1)\pi'(2k)}(n).$$

Therefore, as $\{\sigma_{ij}(n)\}$ is uniformly bounded by M,

$$\begin{aligned}\left|\sum_{\pi\in\Pi^*(w)} \frac{\sigma_\pi(n)}{n^{k+1}} - c^2 \sum_{\pi':\pi\in\Pi^*(w)} \frac{\sigma_{\pi'}(n)}{n^k}\right| &= \left|\sum_{\pi'\in\Pi^*(w')} \frac{\sigma_{\pi'(0)\pi'(1)}(n)\cdots(w_{\pi'(j-1),n})\cdots\sigma_{\pi'(2k-1)\pi'(2k)}(n)}{n^k}\right| \\ &< \epsilon M^{k-1}.\end{aligned}$$

Therefore,

$$\left|\sum_{\pi\in\Pi^*(w)} \frac{\sigma_\pi(n) - c^{2k}}{n^{k+1}}\right| < \epsilon M^{k-1} + c^2 \left|\sum_{\pi'\in\Pi^*(w')} \frac{\sigma_{\pi'}(n) - c^{2k-2}}{n^k}\right|.$$

Hence, by induction, for every $k \geq 1$ and every Catalan word of length $2k$, we get $p(w) = c^{2k}$. Hence the LSD is semicircular with scale c. □

3.2 Proof of Theorem 2

For notational simplicity, we suppress the dependency on n. Moreover, we prove the result for the case when δ_i is a non-random sequence. The case when they are random, follows by a conditioning argument. We proceed by induction:

Induction statement S_k: For each $k \geq 1$, the limit

$$\tau_{2k} \equiv \lim_n \frac{1}{n^{k+1}} \sum_{w \text{ Catalan of length } 2k} \sum_{\pi\in\Pi^*(w)} \delta_{\pi(0)}\sigma_\pi \text{ exists.} \tag{3.4}$$

Consequently,

$$\beta_{2k} = \sum_{p=1}^{k} \sum_{n_1+n_2+\cdots+n_p=k} c_p \prod_{i=1}^{p} \tau_{2n_i-2} \text{ and } \tau_{2k} = \sum_{p=1}^{k} \sum_{n_1+n_2+\cdots+n_p=k} c_{p+1} \prod_{i=1}^{p} \tau_{2n_i-2}.$$

Trivially, for $k = 0$, the limit

$$\tau_0 = \lim_n \frac{1}{n} \sum_i \delta_i = c_1$$

and for $k = 1$, the limit

$$\tau_2 = \lim_n \frac{1}{n^2} \sum_{i,j} \sigma_{ij}^2.\delta_i = \lim_n (\frac{1}{n} \sum_i \delta_i^2)(\frac{1}{n} \sum_i \delta_j) = c_2 c_1$$

exists. Now assume the limit (3.4) exists for $k < t$.We split the rest of the proof into two cases:

(i) Consider a Catalan word $w = aw'a$ of length $2t$ where w' is again a Catalan word of length $2t-2$. If $\pi \in \Pi^*(aw'a)$, then $\pi = (\pi(0)\ \pi'\ \pi(2t))$ where $\pi' \in \Pi^*(w')$, with $\pi(0) = \pi(2t) = i$ and $\pi(1) = \pi(2t-1) = j$ (say). Conversely, for any $1 \leq i \leq n$, any $\pi' \in \Pi^*(w')$, $\pi = (i\ \pi'\ i) \in \Pi^*(aw'a)$.

For this case $\sigma_\pi = \sigma_{ij}^2 \sigma_{\pi'} = \delta_i \delta_j \sigma_{\pi'} = \delta_i \delta_{\pi'(0)} \sigma_{\pi'}$. Now, as the limit

$$\lim_n \frac{1}{n^t} \sum_{\substack{w' \text{ Catalan,} \\ |w|=2t-2}} \sum_{\pi'\in\Pi^*(w')} \delta_{\pi'(0)}\sigma_{\pi'} = \tau_{2t-2} \text{ (say)}$$

exists by induction hypothesis, therefore the following limit also exists:

$$\begin{aligned}
\lim_n \sum_{\substack{w' \text{ Catalan,} \\ |w|=2t-2}} \sum_{\pi\in\Pi^*(aw'a)} \frac{\sigma_\pi}{n^{t+1}} &= \left(\lim_n \frac{1}{n}\sum_{i=1}^n \delta_i\right)\left(\lim_n \frac{1}{n^t} \sum_{\substack{w' \text{Catalan,} \\ |w|=2t-2}} \sum_{\pi'\in\Pi^*(w')} \delta_{\pi'(0)}\sigma_{\pi'}\right) \\
&= c_1\tau_{2t-2}.
\end{aligned}$$

Moreover, we also have

$$\lim_n \frac{1}{n^{t+1}} \sum_{\substack{w' \text{ Catalan,} \\ |w|=2t-2}} \sum_{\pi\in\Pi^*(aw'a)} \delta_{\pi(0)}\sigma_\pi = c_2\tau_{2t-2}.$$

(ii) Now suppose w is of the form $w = (a_1w_1a_1)(a_2w_2a_2)\dots(a_pw_pa_p)$ where each w_i is a Catalan word of length $2n_i - 2$, where $(n_1 + n_2 + \dots + n_p = t)$. Again for each i,

$$\lim_n \frac{1}{n^{n_i}} \sum_{\substack{w_i \text{ Catalan,} \\ |w_i|=2n_i-2}} \sum_{\pi_i\in\Pi^*(w_i)} \delta_{\pi_i(0)}\sigma_{\pi_i} = \tau_{2n_i-2} \text{ exists by induction hypothesis.}$$

Now, $\pi = (\pi(0)\ \pi_1\ \pi(2n_1)\ \pi_2\ \pi(2n_1+2n_2)\ \pi_3\ \dots\pi_p\ \pi(2n_1+2n_2+\dots+2n_p))$, where $\pi_i \in \Pi^*(w_i)$ for each i, and

$$\begin{aligned}
\sigma_\pi &= \sigma_{\pi(0)\pi_1(0)}\cdot\sigma_{\pi_1}\cdot\sigma_{\pi_1(2n_1-2)\pi(2n_1)}\cdot\sigma_{\pi(2n_1)\pi_2(0)}\cdot\sigma_{\pi_2} \\
&\quad \cdots\sigma_{\pi_{p-1}}\cdot\sigma_{\pi_{p-1}(2n_{p-1}-2)\pi(2n_1+2n_2+\cdots 2n_{p-1})}\cdot\sigma_{\pi(2n_1+2n_2+\cdots 2n_{p-1})\pi_p(0)}\cdot\sigma_{\pi_p}\cdot\sigma_{\pi_p(2n_p-2)\pi(2t)}.
\end{aligned}$$

Since $\pi_i(0) = \pi_i(2n_i - 2)$ for all i, and $\pi(0) = \pi(2n_1) = \pi(2n_1 + 2n_2) = \dots = \pi(2t)$, the above expression equals:

$$\sigma_\pi = \delta^p_{\pi(0)}[\delta_{\pi_1(0)}\sigma_{\pi_1}][\delta_{\pi_2(0)}\sigma_{\pi_2}]\cdot[\delta_{\pi_p(0)}\sigma_{\pi_p}].$$

Therefore,

$$\lim_n \frac{1}{n^{t+1}} \sum_{\substack{w=(a_1w_1a_1)(a_2w_2a_2)\dots(a_pw_pa_p): \\ w_i \text{ Catalan,}|w_i|=2n_i-2 \\ \text{for } i=1,\dots,p}} \sum_{\pi\in\Pi^*(w)} \sigma_\pi = (\lim_n \frac{1}{n}\sum_{i=1}^n \delta_i^p)\prod_{i=1}^p \tau_{2n_i-2} = c_p\prod_{i=1}^p \tau_{2n_i-2}.$$

And the following limit also exists:

$$\lim_n \frac{1}{n^{t+1}} \sum_{\substack{w=(a_1w_1a_1)(a_2w_2a_2)\dots(a_pw_pa_p): \\ w_i \text{ Catalan,}|w_i|=2n_i-2 \\ \text{for } i=1,\dots,p}} \sum_{\pi\in\Pi^*(w)} \delta_{\pi(0)}\sigma_\pi = (\lim_n \frac{1}{n}\sum_{i=1}^n \delta_i^{p+1})\prod_{i=1}^p \tau_{2n_i-2} = c_{p+1}\prod_{i=1}^p \tau_{2n_i-2}.$$

Therefore, the existence of the limits (3.4) for all k is proved by induction. Moreover,

$$\beta_{2k} = \sum_{p=1}^k \sum_{n_1+n_2+\cdots+n_p=k} c_p\prod_{i=1}^p \tau_{2n_i-2} \text{ and } \tau_{2k} = \sum_{p=1}^k \sum_{n_1+n_2+\cdots+n_p=k} c_{p+1}\prod_{i=1}^p \tau_{2n_i-2}.$$

The formula (2.4) for $\{\beta_{2k}\}$ is now obtained as follows: For $k = 1$, this is trivial. For $k \geq 2$,

$$\beta_{2k} = c_1\tau_{2k-2} + \sum_{p=2}^k \sum_{n_p=1}^{k-p+1} \left(\sum_{n_1+n_2+\cdots+n_{p-1}=k-n_p} c_p\prod_{i=1}^p \tau_{2n_i-2}\right)$$

$$
\begin{aligned}
&= c_1\tau_{2k-2} + \sum_{t=1}^{k-1}\sum_{p=2}^{k-t+1}\left(\sum_{n_1+n_2+\cdots+n_{p-1}=k-t} c_{(p-1)+1}\prod_{i=1}^{p-1}\tau_{2n_i-2}\right)\tau_{2t-2} \\
&= c_1\tau_{2k-2} + \sum_{t=1}^{k-1}\sum_{p=1}^{k-t}\left(\sum_{n_1+n_2+\cdots n_p=k-t} c_{p+1}\prod_{i=1}^{p}\tau_{2n_i-2}\right)\tau_{2t-2} \\
&= c_1\tau_{2k-2} + \sum_{t=1}^{k-1}\tau_{2k-2t}\tau_{2t-2} = \sum_{t=1}^{k}\tau_{2k-2t}\tau_{2t-2}.
\end{aligned}
$$

The verification of the relation between the generating functions is routine, and is omitted.

Now, we proceed to prove the last part of the theorem. Suppose $c_i = b^i$ for every i. Then

$$\phi_c(x) = \frac{b}{1-bx} \quad \text{and} \quad \phi_\tau(x) = \frac{b}{1-bx^2\phi_\tau(x)}.$$

This implies

$$\phi_\tau(x) = b[1 + (x\phi_\tau(x))^2].$$

Solving this, we get:

$$\phi_\tau(x) = \frac{1-\sqrt{1-4b^2x^2}}{2bx^2}$$

The other solution is eliminated by the continuity of ϕ_τ at 0. Now, from the above equation,

$$
\begin{aligned}
\phi_\beta(x) &= 1 + [x\phi_\tau(x)]^2 \\
&= \frac{\phi_\tau(x)}{b} \\
&= \frac{1-\sqrt{1-4b^2x^2}}{2b^2x^2} \qquad (3.5)
\end{aligned}
$$

which is the well known generating function of the semicircular law with scale parameter b^2.

Conversely, let ϕ_β be of the above form (3.5). Therefore,

$$\phi_\beta(x) = 1 + \left(\frac{1-\sqrt{1-4b^2x^2}}{2bx}\right)^2$$

which implies

$$\phi_\tau(x) = \frac{1-\sqrt{1-4b^2x^2}}{2bx^2}.$$

We know that

$$\phi_c(y) = \frac{y}{x^2} \text{ where } y = x^2\phi_\tau(x).$$

Therefore,

$$1 - 2by = \sqrt{1-4b^2x^2}$$

which implies

$$\frac{y}{x^2} = \frac{b}{1-by}.$$

As a consequence,

$$\phi_c(y) = \frac{b}{1-by}$$

which in turn implies that $c_i = b^i$ for all i.

Hence, all the assertions of the theorem are proved. $\square$

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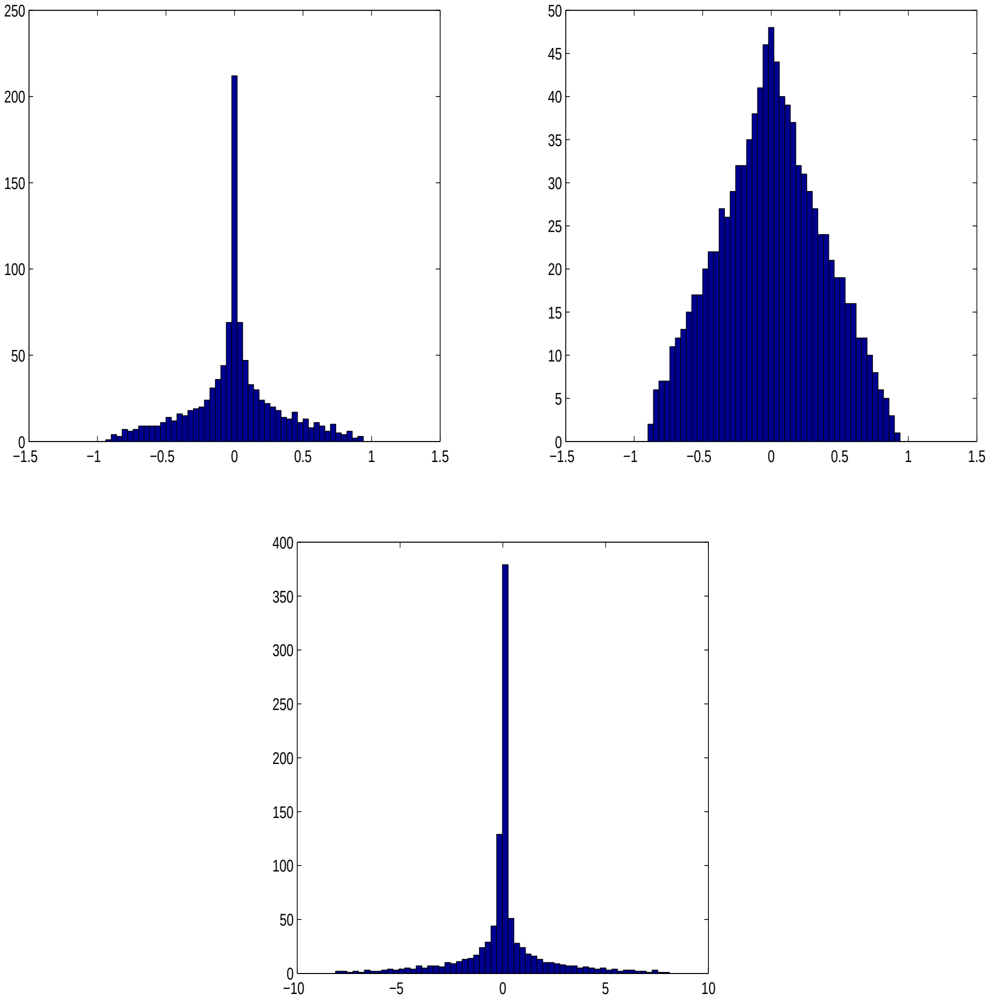

Figure 1: Histogram plots of empirical distribution when x_{ij} are i.i.d. $N(0,\ 1)$ and δ_i are i.i.d. respectively, (i) uniform $(0, 1)$, (ii) scaled binomial $(10, 0.6)$ and (iii) square of standard semicircular law. In all cases, the dimension of the matrix is $n = 1000$