LIMITING SPECTRAL DISTRIBUTION OF CIRCULANT MATRIX WITH
DEPENDENT ENTRIES

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Abstract. In this article, we derive the limiting spectral distribution of the circulant matrix when the input sequence is a stationary infinite order two sided moving average process.

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1. Introduction and Main result

Suppose \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are all the eigenvalues of a square matrix \( A_n \) of order \( n \). Then the empirical spectral distribution function (ESDF) of \( A_n \) is defined as

\[
F_n(x, y) = n^{-1} \sum_{i=1}^{n} I\{\text{Re}\lambda_i \leq x, \text{Im}\lambda_i \leq y\}.
\]

Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of square matrices with the corresponding ESDF \( \{F_n\}_{n=1}^{\infty} \). The Limiting Spectral Distribution (or measure) (LSD) of the sequence is defined as the weak limit of the sequence \( \{F_n\}_{n=1}^{\infty} \), if it exists.

If \( \{A_n\} \) are random, the limit is understood to be in some probabilistic sense, such as “almost surely” or “in probability”. Suppose elements of \( \{A_n\} \) are defined on some probability space \((\Omega, \mathcal{F}, P)\), that is \( \{A_n\} \) are random. Let \( F \) be a nonrandom distribution function. We say the ESD of \( A_n \) converges to the limiting spectral distribution (LSD) \( F \) in \( L_2 \) if at all continuity points \((x, y)\) of \( F \),

\[
\int_{\omega} (F_n(x, y) - F(x, y))^2 dP(\omega) \to 0 \text{ as } n \to \infty
\]

and converges in probability to \( F \) if for every \( \epsilon > 0 \) and at all continuity points \((x, y)\) of \( F \),

\[
P(|F_n(x, y) - F(x, y)| > \epsilon) \to 0 \text{ as } n \to \infty.
\]

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For detailed information on limiting spectral distributions of large dimensional random matrices see \cite{Bai1999} and also \cite{BoseSen2008}.

In this article we focus on obtaining the LSD of the circulant matrix \((C_n)\) given by

\[
C_n = \frac{1}{\sqrt{n}} \begin{bmatrix}
x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\
x_{n-1} & x_0 & x_1 & \cdots & x_{n-3} & x_{n-2} \\
x_{n-2} & x_{n-1} & x_0 & \cdots & x_{n-4} & x_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_0
\end{bmatrix}.
\]

So, the \((i, j)\)th element of the matrix is \(x_{(j-i+n)\mod n}\). The eigenvalues are given by (see for example \cite{BrockwellDavis2002}),

\[
\lambda_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l e^{i\omega_l} = b_k + ic_k \quad \forall \, k = 1, 2, \ldots, n,
\]

where

\[
\omega_k = \frac{2\pi k}{n}, \quad b_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l \cos(\omega_l), \quad c_k = \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} x_l \sin(\omega_l).
\]

The existence of the LSD of \(C_n\) is given by the following theorem of \cite{BoseMitra2002}.

**Theorem 1.1.** Let \(\{x_i\}\) be a sequence of independent random variables with mean 0 and variance 1 and \(\sup_i E |x_i|^3 < \infty\). Then the ESD of \(C_n\) converges in \(L_2\) to the two-dimensional normal distribution given by \(N_2(0, D)\) where \(D\) is a diagonal matrix with diagonal entries \(1/2\).

We investigate the existence of LSD of \(C_n\) under a dependent situation. Let \(\{x_n; n \geq 0\}\) be a two sided moving average process,

\[
x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}
\]

where \(\{a_n; n \in \mathbb{Z}\} \in l_1\), that is \(\sum_n |a_n| < \infty\), are nonrandom and \(\{\epsilon_i; i \in \mathbb{Z}\}\) are iid random variables with mean zero and variance one. We show that the LSD of \(C_n\) continues to exist in this dependent situation. Define \(\gamma_h = \text{Cov}(x_{t+h}, x_t)\). Then it is easy to see that \(\sum_{j \in \mathbb{Z}} |\gamma_j| < \infty\) and the spectral density function of \(\{x_n\}\) is given by

\[
f(\omega) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \gamma_k \exp(ik\omega) = \frac{1}{2\pi} \left[ \gamma_0 + 2 \sum_{k \geq 1} \gamma_k \cos(k\omega) \right]
\text{ for } \omega \in [0, 2\pi].
\]

Let \(f^* = \inf_{\omega \in [0, 2\pi]} f(\omega)\) and \(C_0 = \{\omega \in [0, 2\pi]; f(\omega) = 0\}\). For \(k = 1, 2, \ldots, n\), define

\[
\xi_{2k-1} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \epsilon_t \cos(\omega_k t), \quad \xi_{2k} = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \epsilon_t \sin(\omega_k t).
\]
Define

$$B(\omega) = \begin{pmatrix} a_1(e^{i\omega}) & -a_2(e^{i\omega}) \\ a_2(e^{i\omega}) & a_1(e^{i\omega}) \end{pmatrix},$$

where $a_1(e^{i\omega}) = R[a(e^{i\omega})]$, $a_2(e^{i\omega}) = I[a(e^{i\omega})]$, $a(e^{i\omega})$ is same as defined in Lemma 1.3 and for $z \in \mathbb{C}$, $R(z), I(z)$ denote the real and imaginary part of $z$ respectively. It is easy to see that

$$|a(e^{i\omega})|^2 = a_1(e^{i\omega})^2 + a_2(e^{i\omega})^2 = 2\pi f(\omega).$$

Define for $(x, y) \in \mathbb{R}^2$ and $\omega \in [0, 2\pi]$,

$$H(\omega, x, y) = \begin{cases} P(B(\omega)(N_1, N_2)' \leq \sqrt{2}(x, y)') & \text{if } f(\omega) \neq 0, \\ \mathbb{I}(x \geq 0, y \geq 0) & \text{if } f(\omega) = 0. \end{cases}$$

Since $a(e^{i\omega})$ is continuous on $[0, 2\pi]$, it is easy to verify that for fixed $(x, y)$, $H$ is bounded continuous function in $\omega$. Hence we may define

$$F(x, y) = \int_0^1 H(2\pi s, x, y)ds.$$ F is a proper distribution function.

For any Borel set $B$, let $\lambda(B)$ denote the corresponding Lebesgue measure. It is easy to see that

(i) if $\lambda(C_0) = 0$ then $F$ is continuous everywhere and,

(ii) if $\lambda(C_0) \neq 0$ then $F$ is discontinuous only on $D_1 = \{(x, y) : xy = 0\}$.

**Theorem 1.2.** Suppose $\{\epsilon_i\}$ are iid with $E|\epsilon_i|^{2+\delta} < \infty$. Then the ESD of $C_n$ converges in $L_2$ to the LSD

$$F(x, y) = \int_0^1 H(2\pi s, x, y)ds,$$

and if $\lambda(C_0) = 0$ we have

$$F(x, y) = \iint \mathbb{I}_{(v_1, v_2) \leq (x, y)} \left[ \int_0^1 \mathbb{I}_{(f(2\pi s) \neq 0) \frac{1}{2\pi^2 f(2\pi s)} e^{-\frac{v_1^2 + v_2^2}{2(2\pi s)^2}} ds} \right] dv_1 dv_2.$$

**Remark 1.1.** If $\inf_{\omega \in [0, 2\pi]} f(\omega) > 0$, we can write $F_\omega$ in the following form

$$F(x, y) = \iint \mathbb{I}_{(v_1, v_2) \leq (x, y)} \left[ \int_0^1 \frac{1}{2\pi^2 f(2\pi s)} e^{-\frac{v_1^2 + v_2^2}{2(2\pi s)^2}} ds \right] dv_1 dv_2.$$

**Remark 1.2.** If $\{x_i\}$ are i.i.d, then $f(\omega) = 1/2\pi$ for all $\omega \in [0, 2\pi]$ and the LSD is standard complex normal distribution. This agrees with Theorem 1.1.

Proof of the theorem mainly depends on following two lemmas. Lemma 1.3 follows from [Fan and Yao(2003)] (Theorem 2.14(ii), page 63). For completeness, we have provided a proof. The proof of Lemma 1.4 follows easily from [Bhattacharya and Ranga Rao(1976)] (Corollary 18.3, page 184). We omit the details.
Lemma 1.3. Let \( x_t = \sum_{j=-\infty}^{\infty} a_t \epsilon_{t-j} \) for \( t \geq 0 \), where \( \{\epsilon_t\} \) are i.i.d random variables with mean 0, variance 1 and \( \sum_{j=-\infty}^{\infty} |a_j| < \infty \). Then for \( k = 1, 2, \cdots, n \),

\[
\lambda_k = a(e^{i\omega_k})(\xi_{2k-1} + i\xi_{2k}) + Y_n(\omega_k),
\]

where \( a(e^{i\omega_k}) = \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \) and \( \max_{0 \leq k < n} E[|Y_n(\omega_k)|^2] \to 0 \) as \( n \to \infty \).

Proof.

\[
\lambda_k = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} x_t e^{i\omega_k t} = \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \sum_{t=0}^{n-1} \epsilon_{t-j} e^{i\omega_k (t-j)} = \frac{1}{\sqrt{n}} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} \left( \sum_{t=0}^{n-1} \epsilon_t e^{i\omega_k t} + U_{nj} \right) = a(e^{i\omega_k})(\xi_{2k-1} + i\xi_{2k}) + Y_n(\omega_k),
\]

where

\[
a(e^{i\omega_k}) = \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j}, \quad U_{nj} = \sum_{t=-j}^{n-1-j} \epsilon_t e^{i\omega_k t} - \sum_{t=0}^{n-1} \epsilon_t e^{i\omega_k t}, \quad Y_n(\omega_k) = n^{-1/2} \sum_{j=-\infty}^{\infty} a_j e^{i\omega_k j} U_{nj}.
\]

Note that if \( |j| < n \), \( U_{nj} \) is a sum of \( 2|j| \) independent random variables, whereas if \( |j| \geq n \), \( U_{nj} \) is a sum of \( 2n \) independent random variables. Thus \( E[|U_{nj}|^2] \leq 2 \min(|j|, n) \). Therefore, for any fixed positive integer \( l \) and \( n > l \),

\[
E[|Y_n(\omega_k)|^2] \leq \frac{1}{n} \left( \sum_{j=-\infty}^{\infty} |a_j| (E[U_{nj}^2])^{1/2} \right)^2 \left( \sum_{j=-\infty}^{\infty} |a_j| < \infty \right) \leq \frac{2}{n} \left( \sum_{j=-\infty}^{\infty} |a_j| \min(|j|, n))^{1/2} \right)^2 \leq 2 \left( \frac{1}{\sqrt{n}} \sum_{|j| \leq l} |a_j||j|^{1/2} + \sum_{|j| > l} |a_j| \right)^2.
\]

Note that the right-hand side of the above expression is independent of \( k \) and as \( n \to \infty \), it can be made smaller than any given positive constant by choosing \( l \) large enough. Hence, \( \max_{1 \leq k \leq n} E[|Y_n(\omega_k)|^2] \to 0 \).

Lemma 1.4. Let \( X_1, \ldots, X_k \) be independent random vectors with values in \( \mathbb{R}^d \), having zero means and an average positive-definite covariance matrix \( V_k = k^{-1} \sum_{j=1}^{k} \text{Cov} X_j \). Let \( G_k \)}
denote the distribution of $k^{-1/2}T_k(X_1 + \ldots + X_k)$, where $T_k$ is the symmetric, positive-definite matrix satisfying $T_k^2 = V_k^{-1}$, $n \geq 1$. If for some $\delta > 0$, $E \| X_j \|^{(2+\delta)} < \infty$, then

$$\sup_{C \in \mathcal{C}} |G_k(C) - \Phi_{0,I}(C)| \leq c k^{-\delta/2} \left[ k^{-1} \sum_{j=1}^{k} E \| T_k X_j \|^{(2+\delta)} \right]$$

$$\leq c k^{-\delta/2} (\lambda_{\min}(V_k))^{-(2+\delta)} \left[ k^{-1} \sum_{j=1}^{k} E \| X_j \|^{(2+\delta)} \right]$$

where $\Phi_{0,I}$ is the normal probability function with mean zero and identity covariance matrix, $\mathcal{C}$, the class of all Borel-measurable convex subsets of $\mathbb{R}^d$ and $c$ is a constant, depending only on $d$.

**Proof of Theorem 1.2:** We first assume $\lambda(C_0) = 0$. To prove the theorem it suffices to show that for each $x, y \in \mathbb{R}$,

$$(1.1) \quad E(F_n(x, y)) \to F(x, y) \quad \text{and} \quad V(F_n(x, y)) \to 0.$$ 

Note that we may ignore the eigenvalue $\lambda_n$ and also $\lambda_n/2$ whenever $n$ is even since they contribute at most $2/n$ to the ESD $F_n(x, y)$. So for $x, y \in \mathbb{R}$,

$$E[F_n(x, y)] \sim n^{-1} \sum_{k=1,(k\neq n/2)}^{n-1} P(b_k \leq x, c_k \leq y).$$

Define for $k = 1, 2, \ldots, n$,

$$\eta_k = (\xi_{2k-1}, \xi_{2k})', \quad Y_1\omega_k = \mathcal{R}[Y_\omega(k)], \quad Y_2\omega_k = \mathcal{I}[Y_\omega(k)],$$

$$A_k = \begin{pmatrix} a_1(e^{i\omega_k}) & -a_2(e^{i\omega_k}) \\ a_2(e^{i\omega_k}) & a_1(e^{i\omega_k}) \end{pmatrix},$$

where $a(e^{i\omega_k}), Y_\omega(k)$ are same as defined in Lemma 1.3. Then $(b_k, c_k)' = A_k \eta_k + (Y_1\omega_k, Y_2\omega_k)'$. From Lemma 1.3, it is intuitively clear that for large $n$, $\lambda_k \sim a(e^{i\omega_k})[\xi_{2k-1} + i\xi_{2k}]$. So first we show that for large $n$

$$\frac{1}{n} \sum_{k=1,(k\neq n/2)}^{n-1} P(b_k \leq x, c_k \leq y) \sim \frac{1}{n} \sum_{k=1,(k\neq n/2)}^{n-1} P(A_k \eta_k \leq (x, y)') .$$
Note
\[
\frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(b_k \leq x, c_k \leq y) - \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x,y)')
\]
\[
= \left| \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k \eta_k + (Y_{1n}(\omega_k), Y_{2n}(\omega_k))' \leq (x,y)') - P(A_k \eta_k \leq (x,y)') \right|
\]
\[
\leq \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(|Y_{1n}(\omega_k)|, |Y_{2n}(\omega_k)|) > (\epsilon, \epsilon)
\]
\[
+ \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x,y)',(|Y_{1n}(\omega_k)|, |Y_{2n}(\omega_k)|) \leq (\epsilon, \epsilon)) - P(A_k \eta_k \leq (x,y)')
\]
\[
= T_1 + T_2, \text{ say.}
\]

Now using Lemma 1.3, as \(n \to \infty\)
\[
T_1 \leq \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(|Y_{1n}(\omega_k)|^2 > 2\epsilon^2) \leq \frac{1}{2\epsilon^2} \sup_k E|Y_{1n}(\omega_k)|^2 \to 0.
\]

\[
T_2 \leq \max \left\{ \left| \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x+\epsilon,y+\epsilon)') - P(A_k \eta_k \leq (x,y)') \right|, \right.
\]
\[
\left. \left| \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x-\epsilon,y-\epsilon)') - P(A_k \eta_k \leq (x,y)') \right| \right\}
\]
and
\[
\left| \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x+\epsilon,y+\epsilon)') - P(A_k \eta_k \leq (x,y)') \right| \leq T_3 + T_4 + T_5,
\]

where
\[
T_3 = \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k \xi_k \leq (x,y)') - P(A_k (N_1 N_2)' \leq (\sqrt{2}x, \sqrt{2}y)')
\]
\[
T_4 = \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k \xi_k \leq (x+\epsilon,y+\epsilon)') - P(A_k (N_1 N_2)' \leq (\sqrt{2}x+\sqrt{2}\epsilon, \sqrt{2}y+\sqrt{2}\epsilon)')
\]
\[
T_5 = \frac{1}{n} \sum_{k=1,(k \neq n/2)}^{n-1} P(A_k (N_1 N_2)' \leq (\sqrt{2}x+\sqrt{2}\epsilon, \sqrt{2}y+\sqrt{2}\epsilon)') - P(A_k (N_1 N_2)' \leq (\sqrt{2}x, \sqrt{2}y)').
\]

To show \(T_3, T_4 \to 0\) define for \(k = 1, 2, \cdots, n-1\), (except for \(k = n/2\)) and \(l = 0, 1, 2, \cdots, n-1\),
\[
X_{l,k} = (\sqrt{2}\epsilon_l \cos(\omega_k l), \sqrt{2}\epsilon_l \sin(\omega_k l))'.
\]

Note that
\[
E(X_{l,k}) = 0 \quad \forall \ l, k, n.
\]
\[
(1.3) \quad \sum_{l=0}^{n-1} n^{-1} \text{Cov}(X_{l,k}) = I \quad \forall k, n.
\]

Note that for \( k \neq n/2 \)

\[
\{A_k \eta_k \leq (x, y)'\} = \{A_k (n^{-1/2} \sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)'\}.
\]

Since \( \{(r, s) : A_k (r, s)' \leq (\sqrt{2}x, \sqrt{2}y)'\} \) is a convex set in \( \mathbb{R}^2 \) and \( \{X_{l,k}, l = 0, 1, \ldots (n - 1)\} \) satisfies (1.2) and (1.3), we can apply Lemma 1.4 for \( k \neq n/2 \) to get

\[
|P(A_k (n^{-1/2} \sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)') - P(A_k (N_1, N_2) \leq (\sqrt{2}x, \sqrt{2}y)')| \leq cn^{-\delta/2} [n^{-1} \sum_{l=0}^{n-1} E \| X_{l,k} \|^{(2+\delta)}],
\]

where \( N_1, N_2 \) are independent standard normal variates. Note that

\[
\sup_{1 \leq k \leq n} \left[n^{-1} \sum_{l=0}^{n-1} E \| X_{l,k} \|^{(2+\delta)}\] \leq M < \infty
\]

and, as \( n \to \infty \)

\[
\frac{1}{n} \sum_{k=1, (k \neq n/2)}^{n-1} \left| P(A_k (n^{-1/2} \sum_{l=0}^{n-1} X_{l,k}) \leq (\sqrt{2}x, \sqrt{2}y)') - P(A_k (N_1, N_2) \leq (\sqrt{2}x, \sqrt{2}y)') \right| \leq cM n^{-\delta/2} \to 0.
\]

Hence \( T_3 \to 0 \) and similarly \( T_4 \to 0 \). and also

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1, (k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x, y)') = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1, (k \neq n/2)}^{n-1} H(\frac{2\pi k}{n}, x, y)
\]

\[
= \int_0^1 H(2\pi s, x, y) ds.
\]

Therefore

\[
\lim_{n \to \infty} T_5 = \left| \int_0^1 H(2\pi s, x + \epsilon, y + \epsilon) ds - \int_0^1 H(2\pi s, x, y) ds \right|
\]

\[
\leq \int_0^1 |H(2\pi s, x + \epsilon, y + \epsilon) ds - H(2\pi s, x, y)| ds.
\]

Note that

\[
|H(2\pi s, x + \epsilon, y + \epsilon) ds - H(2\pi s, x, y)| \leq 2
\]

and for fixed \( (x, y) \in \mathbb{R}^2 \) as \( \epsilon \to 0 \),

\[
(1.4) \quad |H(2\pi s, x + \epsilon, y + \epsilon) ds - H(2\pi s, x, y)| \to 0.
\]

Hence by DCT \( \lim_{\epsilon \to 0} \lim_{n \to \infty} T_5 = 0 \) and

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1, (k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x + \epsilon, y + \epsilon)' - P(A_k \eta_k \leq (x, y)') = 0.
\]
Also note that for fixed \((x, y)\) as \(\epsilon \to 0\),
\[
H(2\pi s, x - \epsilon, y - \epsilon)ds - H(2\pi s, x, y) \to 0,
\]
outside the measure zero set \(C_0\). Using this fact, proceeding as above we can show that
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \sum_{k=1, (k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x - \epsilon, y - \epsilon) - P(A_k \eta_k \leq (x, y)) = 0,
\]
and hence \(\lim_{n \to \infty} T_2 = 0\). Therefore as \(n \to \infty\),
\[
E[F_n(x, y)] \sim \frac{1}{n} \sum_{k=1, (k \neq n/2)}^{n-1} P(A_k \eta_k \leq (x, y)) \to \int_0^1 H(2\pi s, x, y)ds,
\]
and since \(\lambda(C_0) = 0\), we have
\[
\int_0^1 H(2\pi s, x, y)ds = \int_0^1 \mathbb{1}_{\{f(2\pi s) \neq 0\}} H(2\pi s, x, y)ds
\]
\[
= \int_0^1 \mathbb{1}_{\{f(2\pi s) \neq 0\}} \left[ \int \int \mathbb{1}_{\{B(2\pi s)(u_1, u_2)^{(x, y)}\}} \frac{1}{2\pi} e^{-\frac{u_1^2 + u_2^2}{2}} du_1 du_2 \right] ds
\]
\[
= \int_0^1 \mathbb{1}_{\{f(2\pi s) \neq 0\}} \left[ \int \int \mathbb{1}_{\{(v_1, v_2) \leq (x, y)\}} \frac{1}{2\pi^2 f(2\pi s)} e^{-\frac{v_1^2 + v_2^2}{2}} dv_1 dv_2 \right] ds
\]
\[
= \int \int \mathbb{1}_{\{(v_1, v_2) \leq (x, y)\}} \left[ \int_0^1 \mathbb{1}_{\{f(2\pi s) \neq 0\}} \frac{1}{2\pi^2 f(2\pi s)} e^{-\frac{v_1^2 + v_2^2}{2}} ds \right] dv_1 dv_2
\]
\[
= F(x, y).
\]
Now, to show \(V[F_n(x, y)] \to 0\), it is enough to show that
\[
(1.6) \quad \frac{1}{n^2} \sum_{k \neq k'; k, k' = 1}^{n} Cov(J_k, J_{k'}) \to 0.
\]
where for \(1 \leq k \leq n\), \(J_k\) is the indicator that \(\{b_k \leq x, c_k \leq y\}\). Observe that
\[
\frac{1}{n^2} \sum_{k \neq k'; k, k' = 1}^{n} Cov(J_k, J_{k'}) = \frac{1}{n^2} \sum_{k \neq k'; k, k' = 1}^{n} [E(J_k, J_{k'}) - E(J_k)E(J_{k'})].
\]
Now as \(n \to \infty\),
\[
\frac{1}{n^2} \sum_{k \neq k'; k, k' = 1}^{n} E(J_k)E(J_{k'}) = \left( \frac{1}{n} \sum_{k=1}^{n} E(J_k) \right)^2 - \frac{1}{n^2} \sum_{k=1}^{n} (E(J_k))^2 \to H(x, y)^2.
\]
So to show (1.6), it is enough to show as \(n \to \infty\),
\[
\frac{1}{n^2} \sum_{k \neq k'; k, k' = 1}^{n} E(J_k, J_{k'}) \to H(x, y)^2.
\]
Along the lines of the proof used to show \(\frac{1}{n} \sum_{k=1}^{n} P(A_k(N_1, N_2)^{'} \leq (\sqrt{2x}, \sqrt{2y})^{'}) \to F(x, y)\),
one may now extend the vectors of two coordinates defined above to ones with four coordinates and proceed exactly as above to verify this. We omit the routine details.
When \( \lambda(C_0) \neq 0 \), we have to show (1.1) only at continuity points of \( F \) and it is continuous on complement of \( D_2 \). All the above steps except (1.4), (1.5) in the proof will go through for all \((x, y)\), but on complement of \( D \) (1.4), (1.5) also holds. Hence if \( \lambda(C_0) \neq 0 \), we have our required LSD. This proves the Theorem.

\[ \square \]

\section*{References}


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