MODEL-THEORETIC GALOIS GROUPS AS QUOTIENTS OF POLISH GROUPS

TOMASZ RZEPECKI

Galois group $\mathrm{Gal}(T)$ of a first order theory T is defined as the quotient of the automorphism group of a monster (e.g. saturated) model $\mathfrak{C}$ by the group generated by pointwise stabilisers of all of its elementary substructures.

One can show that $\mathrm{Gal}(T)$ does not depend on $\mathfrak{C}$ and that it has a canonical topological group structure. However, for a large class of theories, the topology on $\mathrm{Gal}(T)$ is not Hausdorff (it may even be trivial). It turns out that if T is countable, $\mathrm{Gal}(T)$ has additional structure: namely, one can assign to it a Borel cardinality by factoring the quotient map $\mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}(T)$ via type spaces of the form $S_m(M)$ for a countable model M enumerated by m (which are compact Polish), and this Borel cardinality is also canonical.

In recent work with Krzysztof Krupiński, we have shown that if T is countable NIP, then we can find a compact Polish group $\hat{G}$ and an F_σ normal subgroup $H \leq \hat{G}$ which is equivalent to $\mathrm{Gal}(T)$ in both ways, i.e.:

- $\hat{G}/H$ is topologically isomorphic to G (as a topological group),
- $\mathrm{Gal}(T)$ is Borel equivalent to $\hat{G}/H$ (i.e. the Borel cardinality of the relation on $\hat{G}$ of lying in the same coset of H is the same as the Borel cardinality of $\mathrm{Gal}(T)$).

For general theories, we have a weaker result: namely, the first part still holds, and we have that $\hat{G}/H$ has Borel cardinality less than or equal to that of $\mathrm{Gal}(T)$, but we do not know if the opposite can hold as well.

We have analogues for general strong types (on $\mathrm{Aut}(\mathfrak{C})$-orbits) and connected group components of type-definable groups.

This result improves a previous theorem with Krupiński and Pillay ([KPR15]) where we had shown that $\mathrm{Gal}(T)$ is the topological quotient of a compact (possibly non-metrisable) group – we modify some steps in the construction used there (and based on methods developed in [KP14]), and then we develop a new argument to obtain the equal Borel cardinality in the NIP case.

We make substantial use of topological dynamical constructions (enveloping semigroups, ideal groups), and in the NIP case, we use the results of Bourgain, Fremlin and Talagrand.

The theorem yields an arguably more natural proof of equivalence of smoothness and type-definability of strong types (the main result of [KPR15]) and a new estimate on the Borel cardinality of the Galois group (and other strong types).

I will explain the general ideas used in the proof, as well as some corollaries.

References

[KP14] Krzysztof Krupiński and Anand Pillay. *Generalized Bohr compactification and model-theoretic connected components.* (Accepted in Math. Proc. Cambridge). 30th June 2014. arXiv: 1406.7730.

[KPR15] Krzysztof Krupiński, Anand Pillay and Tomasz Rzepecki. *Topological dynamics and the complexity of strong types.* (submitted). 1st Oct. 2015. arXiv: 1510.00340.