

Lecture 3: Principle of inclusion and exclusion

*Instructor: Sushmita Ruj**Scribe: Nitish Kumar Panigrahy and Rahul Anand*

1 Motivation

Among many of the counting techniques, principle of inclusion and exclusion is considered as a basic counting tool. It has a wide area of applications such as counting derangements, counting number of onto functions, counting intersections, Euler's ϕ function, graph coloring, Rook polynomials and many more. Let us understand it with an example.

1.1 Examples

Let us consider a class in which 20 students study Algebra, 25 students study Probability and 8 students study both. If asked how many of the students study either Algebra or Probability, we just tend to add the numbers in both the subjects i.e.

$$20 + 25 = 45$$

But in this we have counted the students who have taken both the courses twice. So we need to subtract them once. Hence the solution is

$$20 + 25 - 8 = 37$$

Clearly for two sets A and B union can be represented as :

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Similarly the principle of inclusion and exclusion becomes more avid in case of 3 sets which is given by :

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

We can generalize the above solution to a set of n properties each having some elements satisfying that property.

2 Generalization for n sets

Theorem 2.1 *In its general form, the principle of inclusion and exclusion states that for finite n sets A_1, \dots, A_n the following holds.*

$$\begin{aligned} \left| \bigcup_{1 \leq i \leq n} A_i \right| &= \sum_{1 \leq i_1 \leq n} |A_{i_1}| - \sum_{1 \leq i_1 \leq i_2 \leq n} |A_{i_1} \cap A_{i_2}| \\ &+ \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{n+1} \left| \bigcap_{i=1}^n A_i \right| \end{aligned}$$

Proof.

Suppose $x \notin A_1 \cup A_2 \dots \cup A_n$. So contribution of x to $A_1 \cup A_2 \dots \cup A_n$ is 0 in L.H.S. In R.H.S also it is 0 as x is not present in any of the sets. So the count becomes 0.

Let $x \in A_1 \cup A_2 \dots \cup A_n$. Clearly for L.H.S the count is 1 as any element will be present only once in a set. Now let us have a look on R.H.S. We need to calculate how many times x occurs on RHS. x will either be present in each individual set or not. Let us assume that x belongs to k such sets from A_1, \dots, A_n . So we have

$\sum_{1 \leq i_1 \leq n} |A_{i_1}| = k$ as x is present in exactly k sets and for others the contribution of x to the sum is 0.

$\sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| = \binom{k}{2}$ as we can choose any two pairs out of k in $\binom{k}{2}$ ways.

$\sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| = \binom{k}{3}$ and so on.

So R.H.S. now becomes

$$\begin{aligned} & \binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \dots + (-1)^{k-1} \binom{k}{k} \\ = & \binom{k}{0} - \binom{k}{0} + \binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \dots + (-1)^{k-1} \binom{k}{k} \\ = & \binom{k}{0} - [\binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \dots + (-1)^k \binom{k}{k}] \\ = & 1 - [1 + (-1)]^k \\ = & 1 \end{aligned}$$

This completes the proof. □

Exercise :

Try to prove the above theorem by principle of induction.

3 Some Useful Applications

3.1 Counting problems

Consider the below problem.

How many positive integers less than 100 is not a factor of 2,3 and 5?

For solving this problem at first we have to find the number of positive integers less than 100 which are divisible by 2 or 3 or 5.

Let A = The set of elements which are divisible by 2

Let B = The set of elements which are divisible by 3

Let C = The set of elements which are divisible by 5

Clearly by principle of inclusion and exclusion :

$$|A \cup B \cup C| = |A| + |B| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$\begin{aligned}
&= \lfloor 99/2 \rfloor + \lfloor 99/3 \rfloor + \lfloor 99/5 \rfloor - \lfloor 99/2 * 3 \rfloor - \lfloor 99/3 * 5 \rfloor - \lfloor 99/2 * 5 \rfloor + \lfloor 99/2 * 3 * 5 \rfloor \\
&= 49+33+19-16-6-9+3 = 73
\end{aligned}$$

So the total number of integers which are not a factor of 2 or 3 or 5 is

$$99 - 73 = 26$$

Observation:

Above mentioned solution can be carried out to count number of primes with in 0 to N . We can find from 0 to N how many of them are not divisible by 2, 3, 5, 7, First we have to select the minimum number of prime numbers such that we can cover all numbers (with in N) divisible by them and adding one more prime does not add to the total cover. Then by principle of inclusion and exclusion we can count the numbers which are not divisible by any of them. For more details the process **Sieve of Erasthones** can be referred.

3.2 Derangements

Problem Statement:

A derangement is a permutation of the elements of 1, 2, 3, . . . n such that none of the elements appear in their original position. Principle of inclusion and exclusion can be used to count number of such derangements among all possible permutaitons.

Solution:

Clearly total number of permutations = $n!$ Now number of ways in which any one of them is at correct position = $\binom{n}{1}(n-1)!$

But by principle of inclusion and exclusion we have included the arrangements in which any two of them has occupied their respective positions twice. So we have to subtract them once.

So number of ways in which any two of them are at correct position is $\binom{n}{2}(n-2)!$ and so on. So the total number of derangements

$$\begin{aligned}
&= n! - [\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \dots + (-1^n)\binom{n}{n}(n-n)!] \\
&= n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1^n)\frac{1}{n!}]
\end{aligned}$$

3.3 Euler’s ϕ function

Interesting fact:

Leonhard introduced the Euler ϕ function in 1763. However, he did not at that time choose any specific symbol to denote it. In a 1784 publication Euler studied the function further, choosing the Greek letter ϕ to denote it.

Definition:

In number theory, Euler’s ϕ function, $\phi(n)$, is an arithmetic function that counts the total number of positive integer which are relatively prime to n and those integer are less than

n . This is also known as *Totient Function*.

Meaning of relatively prime:

A number x is said to be relatively prime to N if $GCD(N, x) = 1$

If N is prime then $\phi(N) = N - 1$

Problem Statement:

Let N be an integer such that $N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_r^{a_r}$ where $p_1, p_2, p_3, \dots, p_r$ are all prime numbers. Find the value of $\phi(N)$.

Solution:

Let $S_i =$ set of integers less than N which are divisible by p_i ($1 \leq i \leq r$)

So $|S_i| = N/p_i$

Hence by applying principle of inclusion and exclusion:

Number of integers which are factors of at least one of the prime numbers is

$$S = \sum_{1 \leq i \leq r} \frac{n}{p_i} - \sum_{1 \leq i, j \leq r} \frac{n}{p_i p_j} + \sum_{1 \leq i, j, k \leq r} \frac{n}{p_i p_j p_k} - \dots$$

Hence by definition

$$\phi(N) = N - S$$

$$= N - \sum_{1 \leq i \leq r} \frac{n}{p_i} + \sum_{1 \leq i, j \leq r} \frac{n}{p_i p_j} - \sum_{1 \leq i, j, k \leq r} \frac{n}{p_i p_j p_k} + \dots$$

$$= N \prod_{i=1}^r \left[1 - \frac{1}{p_i} \right]$$

Assignment:

Prove that

$$\sum_{d|N} \phi(d) = N$$

3.4 Counting the number of elements having at least m properties

Theorem 3.1 Suppose we have a finite n number of sets A_1, \dots, A_n . Let B_m be the set of elements belonging to at least m of A_1, \dots, A_n . Then Prove that

$$|B_m| = \sum_{j=0}^{n-m} (-1)^j \binom{m+j-1}{m-1} S_{m+j} \dots (1)$$

where

$$S_k = \sum_{i_1 \leq i_2 \dots \leq i_k} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|$$

Proof.

Let us prove this by principle of mathematical induction. Clearly by Theorem 2.1 the above equality holds for $m = 1$. Let us assume the above theorem is true for m and we have to prove whether it is true for $m+1$ or not.

So we have to prove

$$|B_{m+1}| = \sum_{j=0}^{n-m-1} (-1)^j \binom{m+j}{m} S_{m+j+1} \cdots (2)$$

Let $c(k, m)$ denote the number of times x belonging to exactly k of A_1, A_2, \dots, A_n is counted on the RHS of (1).

By induction hypothesis,

$$\begin{cases} c(k, m) = 1 & k \geq m \\ c(k, m) = 0 & \text{otherwise} \end{cases}$$

Let us take an element x belonging to exactly k of A_1, A_2, \dots, A_n .

case 1: $k \geq m+1$

Now $c(k, m+1)$ denote the number of times x belonging to exactly k of A_1, A_2, \dots, A_n is counted on the RHS of (2).

$$c(k, m+1) = \sum_{i=0}^{n-m} (-1)^i \binom{m+i}{m} \binom{k}{m+i+1}$$

(As $S_{m+i+1} = \sum |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_{m+i+1}}|$)

Putting $j = i + 1$ we get

$$c(k, m+1) = \sum_{j=1}^{n-m} (-1)^{j-1} \binom{m+j-1}{m} \binom{k}{m+j}$$

But

$$c(k, m) = \sum_{j=0}^{n-m} (-1)^j \binom{m+j-1}{m-1} \binom{k}{m+j} = 1$$

Now $c(k, m) - c(k, m+1)$

$$\begin{aligned} &= \sum_{j=1}^{n-m} (-1)^j \left[\binom{m+j-1}{m-1} + \binom{m+j-1}{m} \right] \binom{k}{m+j} + \binom{k}{m} \\ &= \sum_{j=1}^{n-m} (-1)^j \left[\binom{m+j}{m} \right] \binom{k}{m+j} + \binom{k}{m} \end{aligned}$$

$$\begin{aligned}
&= \binom{k}{m} \sum_{j=1}^{n-m} (-1)^j \left[\binom{k-m}{j} \right] + \binom{k}{m} \\
&= \binom{k}{m} \left[\sum_{j=0}^{n-m} (-1)^j \binom{k-m}{j} \right] \\
&= \binom{k}{m} \left[\sum_{j=0}^{k-m} (-1)^j \binom{k-m}{j} \right] \\
&= \binom{k}{m} (1-1)^{k-m} = 0
\end{aligned}$$

Hence $c(k, m+1) = 1$

case 2: $k \leq m$

$c(k, m+1) = 0$

Because

$$|B_{m+1}| = \sum_{j=1}^{n-m-1} (-1)^j \binom{m+j}{m} S_{m+j+1}$$

Hence By case 1 and case 2

$$\begin{cases} c(k, m+1) = 1 & k \geq m \\ c(k, m+1) = 0 & \text{otherwise} \end{cases}$$

□

4 References

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