Super Resolution Image Reconstruction Through Bregman Iteration using Morphologic Regularization

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Abstract—Multiscale morphological operators are studied extensively in the literature for image processing and feature extraction purpose. In this paper we model a non-linear regularization method based on multiscale morphology for edge-preserving super resolution (SR) image reconstruction. We formulate SR image reconstruction problem as a de-blurring problem and then solve the inverse problem using Bregman iterations. The proposed algorithm can suppress inherent noise generated during low-resolution (LR) image formation as well as during SR image estimation efficiently. Experimental results show the effectiveness of the proposed regularization and reconstruction method for SR image.

Index Terms—Morphologic regularization, Bregman iteration, operator splitting, de-blurring, subgradients.

I. INTRODUCTION

It is always desirable to generate a high resolution (HR) image as it shows more intricate details. However, available sensors have limitation in respect to maximum resolution. Thus our basic goal is to develop an algorithm to enhance the spatial resolution of images captured by an image sensor with fixed resolution. This process is called the Super Resolution (SR) method and remains an active research topic for the last two decades. A number of fundamental assumptions are made about image formation and quality which in turn lead to different SR algorithms. These assumptions include the type of motion, the type of blurring and also the type of noise. It is also to be decided whether to produce the very best HR image possible or an acceptable HR image as quickly as possible. Moreover SR algorithms may vary depending on whether only single LR image is available (single frame SR) or multiple LR images are available (multi-frame SR). Also SR image reconstruction algorithms work either in (a) Frequency domain or in (b) Spatial domain. In this work we focus only on spatial domain approach for multi-frame SR image reconstruction.

Among spatial domain multi-frame SR image reconstruction algorithms, representative works include non-uniform interpolation based approaches [1], [2]. The advantage of these approaches is that their computational cost is relatively low which makes them suitable for real-time applications. However, they do not consider the blur model or noise characteristics, hence quality of the reconstructed HR image is not good. Projections on a convex set (POCS) based methods [3], [4] utilize the spatial domain observation model and incorporate some prior information. Though these methods are simple, their disadvantages are non-uniqueness of solutions, slow convergence rate and heavy computational cost.

Iterative back projection based approaches [5] conduct SR reconstruction in a straightforward way. However, they have no unique solution due to the ill-posed nature of the inverse problem and some parameters are difficult to choose. On the other hand, Bayesian maximum a posteriori (MAP) estimation based methods [6], [7], [8] explicitly use prior information in the form of a prior probability density on HR image and provide a rigorous theoretical framework. In [8], a MAP based joint formulation is proposed that judiciously combine motion estimation, segmentation and SR together.

Another class of SR reconstruction methods generate HR image from a single LR image or frame. These methods are called “Example-Based Super-Resolution” [9], [10], [11], [12] or “image hallucination” [13]. These algorithms are in general based on natural image edge prior [14], [10] or gradient profile prior [15]. Glasner et al. [11] merge the concepts of both single frame SR and multi-frame SR for HR image reconstruction from single frame. In recent years SR image reconstruction algorithms with sparse image prior [12], [13], [16] have been receiving more attention due to advancement of Sparse Coding techniques. Those algorithms are based on dual Dictionary Learning techniques on the pairs of LR and HR image patches.

The approach reported in the present paper is based on the regularization framework, where HR image is estimated based on some prior knowledge about the image (e.g., degree of smoothness) in the form of regularization. The probability based MAP approach, stated earlier is equivalent to the concept of regularization, only in this case it gives a probabilistic meaning to the regularization expression [17]. Tikhonov regularization [18] based on bounded variation (BV) ($L_2$ norm) is one of the popular regularization methods for SR reconstruction. It imposes smoothness in reconstructed image, but at the same time loses some details (e.g., edges) present in the image. First successful edge preserving regularization method for de-noising and de-blurring is total variance (TV) ($L_1$ norm) method [19]. Another interesting algorithm, proposed by Farsiu et al. [20], employs bilateral total variation (BTV) regularization. To achieve further improvement, Li et
al. [21] have used a locally adaptive bilateral total variation (LABTV) operator for the regularization. Recently, two other regularization terms are proposed for SR image reconstruction, viz. “non-local means regularization” (NLM) [22] and “steering kernel regression” (SKR) [23]. Usually, iterative SR image reconstruction methods based on differentiable regularization terms with $L_p$ norm ($1 < p < 2$) [6], [18], [21] use gradient descent approach to obtain optimal solution. On the other hand, quite a few techniques [24], [25], [26] are available in the literature to efficiently handle non-differentiable regularization term with $L_1$ norm (e.g., TV regularization). A group of solvers, evolved from Bregman iteration [27], is one of recently developed methods for such non-differentiable constraint optimization problems. Marquina and Osher [28] are first to use Bregman iteration for fast SR image reconstruction with TV regularization.

However, even though all these regularization terms for SR image reconstruction lead to a stable solution, their performance depends on optimization technique as well as regularization term. For example, with gradient descent optimization technique the LABTV [21] regularization outperforms BTV [20] which gives better result than TV [19] regularization; on the other hand, based on TV regularization Marquina and Osher [28] obtain superior result employing Bregman iteration. So we envisage that even better result would be obtained by combining Bregman iteration and a more sophisticated regularization method that can suppress noise in LR images and ringing artifacts occurred during capturing details of HR image. We propose a new regularization method based on multi-scale morphologic filters which are non-linear in nature. Morphological operators and filters are well-known tool that can extract structure from image [29]. They are used in image denoising [30], [31], image segmentation [32], [33] and image fusion [34], [35] successfully. Since proposed morphologic regularization term uses non-differentiable max and min operators, we develop an algorithm based on Bregman iterations and forward-backward operator splitting using subgradients. It is seen that the results produced due to the proposed regularization are less affected by aforementioned noise evolved during iterative process.

The rest of the paper is organized as follows. Section II introduces the LR image formation model and inverse reconstruction of HR image from multiple LR images with state-of-art regularization methods. We introduce proposed multi-scale morphologic regularization method in Section III. In Section IV, we review Bregman iteration and develop an algorithm for SR image reconstruction with proposed morphologic regularization. Section V presents the experimental results and compares our results with that of some existing regularization methods for SR image reconstruction. Finally, some concluding remarks are made in Section VI.

II. PROBLEM FORMULATION

Observed images of a scene are usually degraded by blurring due to atmospheric turbulence and inappropriate camera settings. The LR images are further degraded due to down-sampling by a factor determined by the intrinsic camera parameters. The relationship between the LR images and the HR image can be formulated as [3], [20], [36], [37]

$$Y_k = DF_k H_k X + e_k, \quad \forall k = 1, 2, ..., K$$  \hspace{1cm} (II.1)

where $Y_k$, $X$ and $e_k$ represent lexicographically ordered column vectors of the $k$-th LR image of size $M$, HR image of size $N$ and additive noise respectively. $F_k$ is a geometric warp matrix and $H_k$ is the blurring matrix of size $N \times N$ incorporating camera lens/CCD blurring as well as atmospheric blurring. $D$ is the down-sampling matrix of size $M \times N$ and $k$ is the index of the LR images. Assuming that the LR images are taken under same environmental condition and using same sensor, the $H_k$ becomes same for all $k$ and may be denoted simply by $H$. As a result, LR images are related to the HR image as

$$Y_k = DF_k HX + e_k, \quad \forall k = 1, 2, ..., K$$  \hspace{1cm} (II.2)

Since under our assumption, $D$ and $H$ are same for all LR images, we avoid down-sampling and then up-sampling at each iteration of iterative reconstruction algorithm [see Section II-A] by merging the up-sampled and shifted back LR images $Y_k$ together. After applying up-sampling and reverse shifting $Y_k$ will be aligned with HR image $X$. Suppose $Y_k$ denote the up-sampled and reverse shifted $k$-th LR image obtained through reverse effect of $DF_k$ of equation (II.2). That means $Y_k = F_k^{-1} D^T Y_k$, where $D^T$ is the up-sampling operator matrix of size $N \times M$ and $F_k^{-1}$ is an $N \times N$ matrix that shifts back (reverse effect of $F_k$) the image. Thus from eqn. II.2 we can write

$$F_k^{-1} D^T Y_k = F_k^{-1} D^T D F_k H X + F_k^{-1} D^T e_k, \quad \forall k = 1, 2, ..., K$$

i.e. $Y_k = R_k H X + e_k, \quad \forall k = 1, 2, ..., K$

where $R_k = F_k^{-1} D^T D F_k$ captures the contribution of pixels of blur image $H X$ from which LR image $Y_k$ is generated. In our formulation we have chosen purely integer valued translational shifts with respect to HR grid. In that case $F_k$ has only 0 and 1 entries. Since $D$ is just a down-sample matrix applied on $F_k$, therefore $R_k$ has only 0 and 1 entries. For more general formulation, if we incorporate rotational shifts as well in our motion model and/or if we use bilinear interpolation for non-integer translational shift, $R_k$ has real entries in the range $[0, 1]$ and we may call it an weight matrix. However, in this work we assume purely integer valued translational shift and $R_k$ is called an index matrix.

Now, suppose $Y$ is obtained by ensembling all the available $Y_k$ into a single HR grid [as shown in figure 1(b)], i.e. $Y = \Sigma_k Y_k$ and $R = \Sigma_k R_k$. Then the relation between HR and LR images can be rewritten as

$$Y = RH X + e$$  \hspace{1cm} (II.3)

Note that the final index matrix $R$ is similar to the transformation matrix derived in [38] except the blurring term. Now, we get $Y$ as a blurred image of actual HR image $X$ except some pixel values may be missing if we take fewer LR images and index matrix $R$ keeps track of those missing values. Thus unknown elements of $Y$ cannot participate in estimating $X$ suggested by eqn. (II.3). Figure 1(a) shows one of four LR images $Y_k$ (i.e., $k = 1, 2, 3, 4$) with resolution factor 5 with
different pixel shifts and figures 1(b)-(c) show corresponding \( X \) and its cropped portion respectively.

\section*{A. L2 error based estimation of SR image}

Since number of unknowns (pixels in HR grid) in \( X \) is usually very large, a solution of equn. (II.3) to obtain \( X \) by inversion of \( RH \) may not be feasible. Instead, we estimate an HR image \( \hat{X} \) which when degraded minimizes \( \rho(RHX, Y) \) i.e. \( \hat{X} = \arg\min X \rho(RHX, Y) \), where dissimilarity measure \( \rho \) is defined as \( \rho(U, V) = \frac{1}{p} ||U - V||^p_p, \) \( (1 \leq p \leq 2) \). Here we choose \( p = 2 \), then the estimated HR image satisfies equation (II.3) in least-square sense, i.e.,

\[ \hat{X} = \arg\min_X \| (RHX - Y) \|_2^2 \] \hspace{1cm} (II.4)

When \( K < N/M \), the SR image reconstruction (II.4) becomes an ill-posed problem, therefore it becomes necessary to impose regularization to obtain a stable solution.

\section*{B. Regularization for SR reconstruction algorithm}

Regularization has already been used in conjunction with iterative methods for restoration of noisy degraded images [18], [20], [21], [39] in order to solve an ill-posed problem and prevent over-fitting. To obtain a stable solution of equn. (II.4), suppose a regularization operator \( \Upsilon(X) \) incorporating prior knowledge is imposed on the estimated HR image \( \hat{X} \). Then the SR image reconstruction can simply be formulated as:

\[ \hat{X} = \arg\min_X \{ \Upsilon(X) : \| (RHX - Y) \|_2^2 < \eta \} \] \hspace{1cm} (II.5)

where \( \eta \) is a scalar constant depending on the noise variance in LR images. Now the constrained minimization problem of (II.5), may be reformulate as an unconstrained minimization problem as

\[ \hat{X} = \arg\min_X \{ \frac{1}{2} \| (RHX - Y) \|_2^2 + \mu \Upsilon(X) \} \] \hspace{1cm} (II.6)

where \( \mu \) is the regularization parameter that controls the emphasis between the data error term (first term) and the regularization term (second term). Choosing regularization parameter \( \mu \) for optimum solution of (II.6) is a non-trivial task. For example, a large value of \( \mu \) may not satisfy the constraint in equn. (II.5) as the data error term is less emphasized, and on the other hand, small value of \( \mu \) may amplify unwanted ringing artifacts as the smoothness criteria gets less importance.

One of the commonly used regularization methods is the Tikhonov cost function [6], [18], [21], [39] for estimating stable de-noised image and is given by

\[ \Upsilon(X) = \| \Gamma X \|_2 \] \hspace{1cm} (II.7)

where \( \Gamma \) is usually a high-pass operator such as derivative or Laplacian. Since high-pass operators enhance both edge and noise equally, minimization of \( \| \Gamma X \|_2 \) leads to blurring of edges along with removing noise. First successful edge preserving regularization method for de-noising and de-blurring is total variation (TV) method [19] which penalizes the spatial variation in the image as measured by the \( L_1 \) norm of the magnitude of the gradient, i.e.,

\[ \Upsilon(X) = \| \Gamma X \|_1 \] \hspace{1cm} (II.8)

Sometimes this is also called ROF (Rudin-Osher-Fatemi) model. Farsiu et al. [20] introduced the bilateral total variation (BTV) by combining the total variation and the bilateral filter [40], [41] as follows.

\[ \Upsilon(X) = \sum_{l=-w}^{w} \sum_{m=-w}^{w} \phi(|X - S^d_m X|) \] \hspace{1cm} (II.9)

where \( l + m \geq 0 \) and \( (S^d_m) \) are shift operator matrices to represent \( l \) and \( m \) pixel shift respectively along horizontal and vertical directions. The parameter \( w \) is the window size and \( \alpha \) \( (0 < \alpha < 1) \) is the weighting coefficient. The amount of edge sharpness preservation as well as the degree of smoothness depend on the value of \( \alpha \) and \( w \).

Thus conventional regularization methods choose \( \Upsilon(X) \) as high-frequency energy and minimize its \( p^2 \) norm to ensure smoothness. In this work we define \( \Upsilon(X) \) based on morpho-logic filters that preserve structures and suppress noise. It is well known that morphological opening and closing removes bright and dark noise respectively without affecting the edge sharpness [29], [30]. An image, in general, may contain both detail and noise at different scales. So we propose a multiscale morphology based regularization to remove noise and artifacts, while preserving the edge sharpness in the next section.

\section*{III. MORPHOLOGIC REGULARIZATION}

Let \( B \) be a disk of unit size with origin at its center and \( sB \) be a disk structuring element (SE) of size \( s \). Then the morphological dilation \( D_s(X) \) of an image \( X \) of size \( m \times n \) at scale \( s \) is defined as:

\[ D_s(X) = \begin{cases} \max_{r \in (sB)_{(1)}} \{ x_r \} \\ \max_{r \in (sB)_{(2)}} \{ x_r \} \\ \vdots \end{cases} \] \hspace{1cm} (III.1)

where \( (sB)_{(i)} \) is a set of pixels covered under SE \( sB \) translated to the \( i \)-th pixel \( x_i \). Similarly morphological erosion \( E_s(X) \) at scale \( s \) is defined as:

\[ E_s(X) = \begin{cases} \min_{r \in (sB)_{(1)}} \{ x_r \} \\ \min_{r \in (sB)_{(2)}} \{ x_r \} \\ \vdots \end{cases} \] \hspace{1cm} (III.2)
Morphological opening \( O_s(X) \) and closing \( C_s(X) \) by SE \( sB \) are defined as:

\[
O_s(X) = D_s(E_s(X)) \\
C_s(X) = E_s(D_s(X))
\]

In multiscale morphological image analysis [30], [33], [42], [43], we have seen that the difference between \( s \)-th scale closing and opening extracts noise particles and image artifacts in scale \( s \) and may be used for de-noising purpose. So in this work we propose the regularization function based on multi-scale morphology as:

\[
\Upsilon(X) = \sum_{s=1}^{S} \alpha^s I^s [C_s(X) - O_s(X)] ~ \text{(III.3)}
\]

where \( I \) is a column vector consisting of all 1’s and \( \alpha \) is the weighting coefficient. To give more emphasis on the small scale for noise removal, the value of \( \alpha \) is chosen from the interval \( 0 < \alpha < 1 \). Therefore, with the proposed regularization term the SR reconstruction problem (II.5) is reduced to:

\[
\hat{X} = \min_X \left\{ \sum_{s=1}^{S} \alpha^s I^s [C_s(X) - O_s(X)] : \|RHX - Y\|_2 < \eta \right\} ~ \text{(III.4)}
\]

Since the proposed regularization term is not differentiable everywhere, we use sub-gradient technique to solve the above reconstruction problem as discussed in the next section.

IV. SUBGRADIENT METHODS AND BREGMAN ITERATION

Bregman iteration is successfully used by Osher et al. [27] in the field of computer vision for finding the optimal value of energy functions in the form of a constrained convex functional. After that a class of efficient solvers has been proposed for constrained problems (II.5) and unconstrained problems (II.6). Among them ‘fixed point continuation’ (FPC) [25] method is proposed to solve the unconstrained problem by performing gradient descent steps iteratively. Linearized Bregman algorithm [44] is derived by combining FPC and Bregman iteration to solve constrained problem in a more efficient way.

Those methods are successfully used in sparse reconstruction problem viz. compressed sensing (CS) [24], [25], [44], [45], [46] and sparse coding [21], [47], [48], [49] due to their simplicity, efficiency and stability. Later Goldstein et al. [50] have developed ‘split Bregman method’ for more structured regularization in variational problems of image processing. Marquina and Osher [28] have formulated a model for SR based on a constrained variational model that uses the total variation of the signal as a regularizing functional. In this section, we develop an algorithm based on Bregman iteration and proposed morphologic regularization for SR image reconstruction problem.

A. Bregman iteration

Consider the following minimization problem:

\[
\min_X \{ \Upsilon(X) : T(X) = 0 \} ~ \text{(IV.1)}
\]

where \( \Upsilon \) and \( T \) are both convex functionals defined over \( \mathbb{R}^n \rightarrow \mathbb{R}^+ \). Now the Bregman iterations [27], [28], [50] that solve the above constrained minimization problem are as follows:

\[
\begin{align*}
\text{Initialize } & X^0 = \rho^0 = 0 \\
\{X^{(n+1)}, \rho^{(n+1)}\} & = \arg \min_{X, \rho} \left\{ \mu \Upsilon(X) + \frac{1}{2} \|T(X)\|_2^2 \right\} \\
\end{align*}
\]

(IV.2)

where \( \mu \) is the Bregman distance corresponding to convex functional \( \Upsilon(.) \) and is defined from point \( X \) to point \( V \) as \( \mu(X;V) = \Upsilon(X) - \Upsilon(V) - \langle p, X - V \rangle \). Convergence and stability of this scheme are discussed in detail in [27], [50].

Yin et al. [44] have shown that for the case of linear constraints \( RHX - Y = 0 \) in (III.4), Bregman iterations (IV.2) can be reduced to a more simplified form [50] with \( l_2 \) norm

\[
\begin{align*}
\{X^{(n+1)}, \rho^{(n+1)}\} & = \arg \min_{X, \rho} \left\{ \mu \Upsilon(X) + \frac{1}{2} \|RHX - Y\|_2^2 \right\} \\
\end{align*}
\]

(IV.3)

In other words the error \( \|Y - RHX^{(n)}\|_2 \) in \( n \)-th estimation is added back to \( Y^{(n)} \) such that finally \( RHX^{(n)} \) satisfies the constraint \( RHX - Y = 0 \). Note that the first equation solve the unconstrained minimization problem (II.6). As, in general, there is no explicit expression for \( X^{(n+1)} \) to solve the unconstrained optimization subproblem (first equation) (IV.3), we go further to solve it explicitly.

B. Proximal map

Consider the following unconstrained minimization problem:

\[
\min_X \{ \mu \Upsilon(X) + T(X) \} \quad \text{(IV.4)}
\]

where \( \mu > 0 \). Combettes et. al. [51] describe a forward-backward technique to minimize the sum of two convex functionals based on the proximal operator introduced by Moreau [52]. By classical arguments of convex analysis, the solution of (IV.4) satisfies the condition:

\[
\mu \partial \Upsilon(X) + \partial T(X) = 0
\]

For any positive number \( \gamma \), we have

\[
\gamma \mu \partial \Upsilon(X) - \partial \gamma T(X) = 0
\]

This leads to a forward and backward splitting algorithm:

\[
X^{(k+1)} = \text{Prox}_\gamma \Upsilon(X^{(k)}) - \gamma \partial T(X^{(k)}) \quad \text{(IV.5)}
\]

where the proximal operator \( \text{Prox}_\gamma \Upsilon(V) \) is defined as:

\[
\text{Prox}_\gamma \Upsilon(V) = \arg \min_X \{ \Upsilon(X) + \frac{1}{2 \gamma} \|X - V\|_2^2 \} \quad \text{(IV.6)}
\]

In our case \( T(X) = \frac{1}{2} \|RHX - Y\|_2^2 \). Therefore the solution of the minimization problem (IV.4) can be computed by the following two-step algorithm:

\[
\begin{align*}
\{U^{(k+1)}, \rho^{(k+1)}\} & = \arg \min_{U, \rho} \left\{ \mu \Upsilon(U) + \frac{1}{2 \gamma} \|U - \rho \|_2^2 \right\} \\
\{X^{(k+1)}, \rho^{(k+1)}\} & = \arg \min_{X, \rho} \left\{ \mu \Upsilon(X) + \frac{1}{2 \gamma} \|X - U^{(k+1)}\|_2^2 \right\}
\end{align*}
\]

(IV.7)

This algorithm (IV.7) can be used to solve the unconstrained minimization problem presented by first equation of (IV.3) as described next.
C. Bregmanized Operator Splitting

In this section we develop an algorithm to solve the equality constrained minimization problem (IV.1) with $T(X) := RHX - Y$ using Bregman iterations (IV.3) and operator splitting (IV.7), briefly introduced in the last two subsections. Now, the operator splitting technique is used to solve the unconstrained subproblem, i.e. first equation of (IV.3) as follows:

$$
U(n+1) = \arg\min_{X} \left\{ \mu Y(X) + \frac{1}{2\gamma} \|X - U(n)\|^2 \right\} \tag{IV.8}
$$

Ideally we need to iterate (IV.8) infinite number of times [53] to obtain a convergent solution $X(n+1)$ for the said subproblem. Instead we propose to iterate them once for small amount of noise, which leads to the following algorithm:

$$
\begin{align*}
U(n+1) &= X(n) - \gamma H^T R^T (RHX(n) - Y^n) \\
X(n+1) &= \arg\min_{X} \left\{ \mu Y(X) + \frac{1}{2\gamma} \|X - U(n+1)\|^2 \right\} \\
Y(n+1) &= Y(n) + (Y - RHX(n+1))
\end{align*} \tag{IV.9}
$$

For optimal solution of the second equation of (IV.9), we have (replacing gradients by its corresponding subgradients)

$$
\mu \frac{\delta Y(X)}{\delta X} + \frac{1}{\gamma} (X - U(n+1)) = 0
$$

So the corresponding iteration can be written as

$$
X(n+1) = U(n+1) - \mu' \left| \frac{\delta Y(X)}{\delta X} \right| X(n)
$$

where $\mu' = \mu \gamma$ and $\left| \frac{\delta Y(X)}{\delta X} \right| X(n)$ is a subgradient of $Y(X)$ at $X(n)$.

Goldstein and Osher [50] have shown that the above algorithm (IV.9) can also be used to approximate the minimizer for linear inequality constraint of the form $\|RHX - Y\|^2 < \eta$ for small value of $\eta$ by imposing a stopping criterion based on inequality constraint that leads to an algorithm for solving inequality constrained minimization (II.5). Therefore we can present the proposed SR image reconstruction algorithm for small amount of noise using Bregman iteration and operator splitting as follows:

**Proposed Iterative Algorithm for SR Image Reconstruction:**

Initialize $Y(0) = n = 0$, $X(0) = \text{FillUnknown}(Y)$;

While($\|RHX(n) - Y\|^2 > \eta$)

$$
\begin{align*}
U(n+1) &= X(n) - \gamma H^T R^T (RHX(n) - Y^n) \\
X(n+1) &= U(n+1) - \mu \left| \frac{\delta Y(X)}{\delta X} \right| X(n) \\
Y(n+1) &= Y(n) + (Y - RHX(n+1))
\end{align*} \tag{IV.10}
$$

$n = n + 1$

end

FillUnknown($Y$) in the initialization step of the above algorithm fills the unknown pixels in $Y$ by the corresponding known neighboring pixels. The parameter $\eta$ is a predefined threshold chosen depending on variance of noise in LR images and significantly small for noise-free LR images. So we iterate until constraint in (II.5) is satisfied.

Note that, for LR images with high noise to satisfy inequality constraint in (II.5), a fraction $\beta$ where $\beta < 1$ of the error $(Y - RHX(n))$ in $n$-th estimation is added back to $Y(n)$, i.e. last equation of (IV.10) is replaced by $Y(n+1) = Y(n) + \beta (Y - RHX(n+1))$ and is iterated once after every $l$ iterations of first two equation.

Since proposed regularization function $Y(X)$ as defined in equn. (III.3) consists of non-differentiable max and min operators $(D_s(X), E_s(X))$, we go for computing subgradients $\frac{\delta Y(X)}{\delta X}$ of the morphological regularization $Y(X)$ in the following subsection.

D. Subgradients of Morphologic Regularization function

Here we derive the subgradients of the dilated (III.1) and eroded (III.2) image with respect to its pixel values. Let us denote the subgradient of a dilated image $D_s(X)$ by $\frac{\delta D_s}{\delta X}$. From the definition of $D_s(X)$, as given in (III.1), $\delta^h$ element of $\delta^h$ column of the subgradient $\frac{\delta D_s}{\delta X}$ is:

$$
\begin{align*}
\delta D_s_{,j} &= \begin{cases} 1 & \text{if } x_i = \max_{r \in \beta(s)} \{ x_r \} \\
0 & \text{if } x_i < \max_{r \in \beta(s)} \{ x_r \} \\
\in [0,1] & \text{elsewhere}
\end{cases} \tag{IV.11}
\end{align*}
$$

Similarly subgradient of an eroded image $E_s(X)$ can be written as follows:

$$
\begin{align*}
\delta E_s_{,j} &= \begin{cases} 1 & \text{if } x_i = \min_{r \in \beta(s)} \{ x_r \} \\
0 & \text{if } x_i > \min_{r \in \beta(s)} \{ x_r \} \\
\in [0,1] & \text{elsewhere}
\end{cases} \tag{IV.12}
\end{align*}
$$

Now in (IV.11) and (IV.12) if we choose subgradient equal to 1 out of the range [0,1]. Then the subgradients become:

$$
\begin{align*}
\delta D_s_{,j} &= \begin{cases} 1 & \text{if } x_i = \max_{r \in \beta(s)} \{ x_r \} \\
0 & \text{if } x_i < \max_{r \in \beta(s)} \{ x_r \} \\
\in [0,1] & \text{elsewhere}
\end{cases} \tag{IV.13}
\end{align*}
$$

$$
\begin{align*}
\delta E_s_{,j} &= \begin{cases} 1 & \text{if } x_i = \min_{r \in \beta(s)} \{ x_r \} \\
0 & \text{if } x_i > \min_{r \in \beta(s)} \{ x_r \} \\
\in [0,1] & \text{elsewhere}
\end{cases} \tag{IV.14}
\end{align*}
$$

Since analogous chain rule holds for subgradients, hence we can write down the sub-gradients of the regularization function $Y(X)$ (III.3) as

$$
\begin{align*}
\frac{\delta Y(X)}{\delta X} &= \frac{\delta}{\delta X} \left\{ \sum_{k=1}^{K} \alpha^k \left[ C_k(X) - O_k(X) \right] \right\} \\
&= \frac{\delta}{\delta X} \left\{ \sum_{k=1}^{K} \alpha^k \left[ C_{k}(X) - O_{k}(X) \right] \right\} \\
&= \frac{\delta}{\delta X} \left\{ \sum_{k=1}^{K} \alpha^k \left[ \delta_{s} D_{s}(X) - \delta_{s} E_{s}(X) \right] \right\} \\
&= \frac{\delta}{\delta X} \left\{ \delta_{s} D_{s}(X) - \delta_{s} E_{s}(X) \right\} \\
&= \frac{\delta}{\delta X} \left\{ \delta_{s} D_{s}(X) - \delta_{s} E_{s}(X) \right\} \\
&= \frac{\delta}{\delta X} \left\{ \delta_{s} D_{s}(X) - \delta_{s} E_{s}(X) \right\} \tag{IV.15}
\end{align*}
$$

Clearly, $\frac{\delta}{\delta D_{s}(X)} E_{s}(D_{s}(X))$ and $\frac{\delta}{\delta E_{s}(X)} D_{s}(E_{s}(X))$ can be calculated following (IV.13) and (IV.14) with respect to $s^{th}$
scale dilation $D_\delta(X) = [d_{s1}, d_{s2}, \ldots, d_{sm}]$ and erosion $E_\delta(X) = [e_{s1}, e_{s2}, \ldots, e_{sm}]$ respectively as follows:

$$E_{s,j} := \frac{\delta E_{s,j}}{\delta s_{ij}} = \begin{cases} 1 & \text{if } d_{s,j} = \min_{r \in (SB)} \{s_{ij}\} \\ 0 & \text{if } d_{s,j} > \min_{r \in (SB)} \{s_{ij}\} \end{cases}$$

(IV.16)

$$D_{s,j} := \frac{\delta D_{s,j}}{\delta s_{ij}} = \begin{cases} 1 & \text{if } e_{s,j} = \max_{r \in (SB)} \{e_{ij}\} \\ 0 & \text{if } e_{s,j} < \max_{r \in (SB)} \{e_{ij}\} \end{cases}$$

(IV.17)

Substituting eqn. (IV.14), (IV.13), (IV.16) and (IV.17) in eqn. (IV.15), we compute the subgradient $\frac{\delta Y(X)}{\delta X}$ which is required in computing third step of (IV.10). In Appendix we discuss a method to compute subgradients of $D_{s}(X)$ and $E_{s}(X)$ using chain rule efficiently.

V. EXPERIMENTAL RESULTS

In this section we study and analyze the performance of the proposed method as well as that of some other existing SR reconstruction methods. In this discussion the proposed algorithm (IV.10) is referred to as ‘Breg + Morph’, while the other methods are referred to as ‘Grd + TV’ [19], ‘Grd + BTV’ [20], ‘Breg + TV’ [28] and ‘Grd + LABTV’ [21] respectively. Here ‘Grd’ stands for gradient descent technique and ‘Breg’ stands for Bregman iteration technique for optimization. ‘Morph’, ‘TV’, ‘BTV’ and ‘LABTV’ stands for morphological, total variation, bi-lateral total variation and locally adaptive bi-lateral total variation regularization respectively. Note that the methods associated with ‘Breg’ solve constrained SR reconstruction formulation (II.5) whereas those with ‘Grd’ solve the unconstrained formulation (II.6).

Experimental Setting: For performance evaluation a typical $512 \times 512$ gray-level HR image (e.g., Chart image) is chosen [see figure 2(a)]. We synthetically generate some LR images from this HR image and later reconstruct a HR image from these generated LR images. Finally, we compute PSNR and SSIM as quantitative measure of quality of reconstructed HR image with respect to the original HR image. The blurring is chosen as 5 x 5 Gaussian smoothing kernel with scale parameter $\sigma = 2.5$ and the matrix $H$ is formed accordingly. The down-sampling factor is chosen to be 5 and the matrix $D$ is constructed to achieve this. We assume purely translational shifts by integer value and $F_k$ is formed accordingly. There are 25 possible choices of integer shifts for resolution factor 5 to generate 25 LR images. In our experiment we have observed that only 10 LR images among 25 are sufficient to reconstruct HR image of acceptable quality with resolution factor 5. Hence, in our experiment we take 10 randomly chosen LR images and determine the index matrix $R$ of eqn. (II.3) accordingly. The same setup is followed throughout this experimental section unless stated otherwise. Second, an interesting portion is marked on each resultant HR image and is displayed on the top-right/left corner after zooming it for careful study of the visual quality. The parameters for each algorithm are chosen to maximize the PSNR with respect to the ground truth. In our reconstruction algorithm (IV.10), we have chosen the model parameters $\gamma = 1$ and $\mu = 0.5$ respectively and have got good results in most of the experiments. For the experiments with noisy LR images after every 5 iterations of the first two steps of the algorithm (IV.10) the third step is executed once and $\beta$ is chosen as reciprocal of the noise variance. We have implemented the algorithms using MATLAB 7.6 and run on a regular Desktop with 3Gz Intel quad-core processor and 8GB of RAM.

In the first experiment, LR images contain small amount of noise ($\sigma = 2$) and the up-sampled-merged image obtained from 10 LR images is shown in figure 2(c). The results of different algorithms are shown in figures 2(d)-(h). Figures 2(d)-(f) present SR reconstructed images using gradient descent method for optimization with TV, BTV and LABTV regularizations respectively. Similarly figures 2(g)-(h) present the results using Bregman iteration method for optimization with TV and morphologic regularizations respectively. In each case algorithm is terminated if the residue is less than certain threshold (e.g. $\eta$ in the algorithm (IV.10)) or number of iterations is more than 1000. In figure 3(a) we plot how different algorithms approach the terminating condition. Table I shows the number of iterations and corresponding numerical time comparison of the results shown in figure 2. It is seen that the morphologic regularization yields SR reconstructed image of better quality compared to other regularization methods with less number of iterations. In section IV-A it is mentioned that the convergence and stability of Bregman iteration based scheme are discussed in detail in [27], [50]. However, here we give an indication of the same based on experimental data. For this purpose in Figure 3(b) we plot the objective function of ‘Breg+TV’ and ‘Breg+Morph’ up to the number of iterations as given in Table I. The figure shows that after initial irregularities values of the objective function follows overall a decreasing in nature. Caption of figure 2 provides the quantitative measure of quality of the methods. In Figure 3(c), we plot improvement in PSNR of the estimated image $X^{(n)}$ versus $n$ for different SR reconstruction methods. These two plots, viz. Figures 3(a) and 3(c), together show the reconstruction qualities of different methods with number of iterations. Hence, with a chosen bound on residue as the stopping criterion for SR algorithms, the proposed method can achieve better quality with less number of iterations.

We have applied the proposed SR reconstruction and also other methods over some more images and the results of only man, boat and Lena images are shown in figure 4. In each of the images the proposed method reconstructs more detail than other existing state-of-art reconstruction methods and we also achieve higher PSNR and SSIM in each case.

In the next experiment, we add high Gaussian noise with zero mean and standard deviation $\sigma = 12.0$ to the LR images. The results of different approaches are shown in figure 5. Here
Fig. 2. Illustrates results of various SR image reconstruction methods with small amount of noise ($\sigma = 2$): (a) Original HR image of a Chart, (b) One of the generated LR images, (c) Up-sampled and merged 10 LR images, (d)-(f) SR reconstructed image using gradient descent method with TV, BTV and LABTV regularization respectively, (g)-(h) SR reconstructed image using Bregman Iteration method with TV and morphologic regularization respectively. Number of iterations in each case are shown in Table I.

Fig. 3. Comparison of reconstruction qualities of different methods versus number of iterations for the experiment in figure 2. (a) Illustrates how residue of data fidelity term approaches the threshold value to terminate the algorithm IV-A, (b) variation in objective function with iteration for 'Breg+TV' and 'Breg+Morph' [see text] and (c) PSNR up to 100 iterations for different algorithms as indicated in (a). Actual number of iterations and run time are shown in Table I.

Fig. 4. Comparison of reconstruction result of (a) man, (b) boat and (c) Lena images of proposed method over some existing methods: (i) SR image reconstructed using 'Grd + BTV' [20], (ii) SR image reconstructed using 'Breg + TV' [28], and (ii) SR image reconstructed using 'Breg + Morph' (Proposed method). we see that proposed method is comparable to best performing methods 'Grd + BTV' [20] and 'Grd + LABTV' [21] in terms of both PSNR and SSIM.

In case of salt-and-pepper noise or impulse noise with random values, we can identify the position of noisy pixels by employing the concept of Center-Weighted Median Filters (CWM) [54], [55] filters. So that such blur and noisy LR images may be handled by a two-phase algorithm. In the first phase, we find the locations of the pixels affected by noise, and in the second phase, those pixels are ignored for reconstruction of HR image. In the next experiment we consider LR images having 10% impulse noise with uniform distribution in the range $[-128, 128]$. Now CWM replaces the noisy pixel with the median value of the neighboring pixels and keeps the unaffected pixel as it is. The search for the noisy pixels using CWM filter depends on the neighborhood statistics of the pixels. Since the neighboring pixels of a pixel in a LR image does not remain neighboring ones when they are up-sampled and ensembled into an HR image $\mathbf{Y}$ (see figure 1), we consider required number of closest known neighboring pixels in $\mathbf{Y}$ of the candidate pixel. However, unlike [54], [55] in this work after detecting the noisy pixels we do not modify their values, rather we mark them as unknown pixels and modify the
of the proposed reconstruction method under miss-estimation of motion parameter and also the parameter of Gaussian blur. We corrupt the actual shifts of LR images by adding Gaussian distributed random values with zero mean and standard deviation 0.1 provided the changed values lie within the allowable range. For example, in case of resolution factor 5, the allowable range of shifts is [0, 5). If a noisy shift goes beyond this range it may be either truncated to lie within this range or a new random value is selected to corrupt the shift value. Here we adopt the later approach. However, this choice is not critical. Now the matrix $F_k^{-1}$ is formed based on these noisy shifts and the corresponding matrix $R$ of eqn. (II.3) would no longer be an index matrix rather an weight matrix. Moreover, though we generate the LR images using Gaussian blur with $\sigma = 2.5$, while reconstructing HR image we consider $\sigma = 2.35$. Figure 7 shows the results of this experiment. From the experiment it is seen that even under mispredicted motion parameter and scale of Gaussian blur the proposed SR reconstruction gives comparable result over the existing methods in terms of both PSNR and SSIM.

A more systematic study of performance of different algorithms for different amount of noise and different blurring parameter is conducted. In Figure 8, we plot the average PSNR and SSIM of all the methods mentioned earlier applied on different images. In the experiment we have added different amount of Gaussian noise (standard deviation $\sigma$ = 0 to 10) to LR images and applied various SR reconstruction algorithms. This is done on a set of images and then average PSNR and average SSIM are plotted. We observe that most of the cases proposed method is superior to the existing methods. Only exception is that with large amount of noise ‘Grd + BTV’ and ‘Grd + LABTV’ perform slightly better than the proposed method. Same experiment is done for varying blur parameter in reconstruction algorithms also. Here the blurring parameter in reconstruction model is varied from 1 to 4, while actual blurring parameter is 2.5. It is seen that the proposed method performs the best and the performance of all the methods is peaked at $\sigma = 2.5$ and falls off on either side.

VI. CONCLUSION

In this paper, we have presented an edge-preserving SR image reconstruction problem as de-blurring problem (II.3) with a new robust morphologic regularization method. Then
we put forward two major contributions. First, we have proposed a morphologic regularization function based on multiscale opening and closing which can remove noise efficiently while preserving edge information. Secondly, we employ Bregman iteration method to solve the inverse problem for SR reconstruction with proposed morphologic regularization. It is well studied that multi-scale morphological filtering can reduce noise efficiently, so we have built up successfully a regularization method based on multiscale morphology and our experimental section shows that it works quite well, in fact better than existing methods. Non-linearity of the regularization function is handled in a linear fashion during optimization by means of sub-gradient and proximal map concept.

We also showed that if there are ‘impulse noise with random values’ or ‘salt-and-pepper’ noise in LR images, they can be handled efficiently using our two-step SR reconstruction algorithm. It first detects the noisy pixels (note: it does not substitute their values) and then considering those detected pixels as unknown pixels, it reconstructs SR image using only those pixels which are not corrupted by noise.

The morphologic regularization method proposed here is tested only on SR reconstruction problem, but one can easily extend this work to other ill-posed problem as well. Also one can extend this regularization method to be adaptive by choosing SE of different shapes and sizes depending on the local statistics of neighboring pixels.

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APPENDIX

A. Subgradients of Morphological operators

Let us consider the error \( Y(X) \) due to regularization as mentioned in (III.3). Now use the following properties of Morphological dilation and erosion:

\[
D_y(X) = (X \ominus B) \oplus s \text{ times } B = D(D(\ldots s \text{ times } \ldots D(X) \ldots ))
\]

\[
E_y(X) = (X \ominus B) \oplus s \text{ times } B = E(E(\ldots s \text{ times } \ldots E(X) \ldots ))
\]

where we use simply the operator \( D(X) \) for morphological dilation instead of \( D_1(X) \) and \( E(X) \) instead of \( E_1(X) \).

Since an analogous chain rule holds, subgradient of the regularizing error \( E(X) \) described in (III.3) can be obtained using the chain rule of subgradient and the above properties of morphological operators as follows:

\[
\frac{\delta}{\delta X} D_y(X) = \frac{\delta}{\delta X} [D(D_{s-1}(X))] = \frac{\delta D_{s}}{\delta D_{s-1}} \frac{\delta D_{s-1}}{\delta D_{s-2}} \ldots \frac{\delta D_{1}}{\delta X} \quad (A1)
\]

\[
\frac{\delta}{\delta X} E_y(X) = \frac{\delta}{\delta X} [E(E_{s-1}(X))] = \frac{\delta E_{s}}{\delta E_{s-1}} \frac{\delta E_{s-1}}{\delta E_{s-2}} \ldots \frac{\delta E_{1}}{\delta X} \quad (A2)
\]

Since computing morphological filters \( [IV.13], [IV.14] \) are just computing maximum and minimum in a neighborhood defined by structuring element \( sB \), as scale \( s \) increases, the size of morphological operator \( sB \) also increases and as a result computational time for maximum and minimum values also increases. We use the above chain rule (A1) and (A2) to reduce computation searching for large neighborhood. So we search in unit scale and perform searching \( s-t \) times on \( i^{th} \) subgradient of dilated or eroded image. Let us denote \( Z_1^d = \frac{\delta D_1}{\delta X} \) and \( Z_1^e = \frac{\delta E_1}{\delta X} \) are column vectors obtained from (IV.13) and (IV.14) as follows:

\[
Z_1^d = [\frac{\delta D_1}{\delta X} \frac{\delta D_1}{\delta X} \ldots \frac{\delta D_{mn}}{\delta X} ]^T = z_j \in B(i) \text{ Such that } x_j = \max_{s \in B(j)} \{ x_s \}
\]

\[
Z_1^e = [\frac{\delta E_1}{\delta X} \frac{\delta E_1}{\delta X} \ldots \frac{\delta E_{mn}}{\delta X} ]^T = z_j \in B(i) \text{ Such that } x_j = \min_{s \in B(j)} \{ x_s \}
\]

Therefore following the above chain rule (A1) and (A2) we get:

\[
\frac{\delta}{\delta X} [D_2(X)] = \frac{\delta D_2}{\delta X} Z_1^d
\]

\[
\frac{\delta}{\delta X} [E_2(X)] = \frac{\delta E_2}{\delta X} Z_1^e
\]
Thus we get recursive expression

\[ Z^{ds} := \frac{\delta}{\delta X} [D_s(X)]|1 = \frac{\delta D_s}{\delta D_{s-1}} Z^{d_{s-1}} \]

\[ = \frac{\delta D_{s-1}}{\delta (D_{s-1})^2} Z^{d_{s-1}} \]

\[ Z^{es} := \frac{\delta}{\delta X} [E_s(X)]|1 = \frac{\delta E_s}{\delta E_{s-1}} Z^{e_{s-1}} \]

\[ = \frac{\delta E_{s-1}}{\delta (E_{s-1})^2} Z^{e_{s-1}} \]

Where \( \frac{\delta D_s(D_{s-1})}{\delta D_{s-1}} \) and \( \frac{\delta E_s(E_{s-1})}{\delta E_{s-1}} \) are computed in the same way as in equations (IV.13) and (IV.14) with respect to dilation \( D_{s-1} \) and erosion \( E_{s-1} \) respectively with unit scale.

B. Computational Complexity for proposed Morphologic Regularization term

We use the above method for efficient computation of subgradients of morphological operators. Now computing \( \frac{\delta}{\delta X} D_s(X) \) in (A1) and \( \frac{\delta}{\delta X} E_s(X) \) in (A2) as above would take the same number of operations as derivatives in ‘BTV’ regularization term (II.9) for \( w = s \) and \( 2/(2s+1)^2 \) comparison that ‘TV’ regularization. Since in ‘TV’ regularization term we need to compute only first order derivatives and for ‘BTV’ regularization higher order derivatives for all neighborhood pixels. Now computing \( \frac{\delta}{\delta D_s(X)} E_s(D_s(X)) \) and \( \frac{\delta}{\delta E_s(X)} D_s(E_s(X)) \) as shown in equn. (IV.16) and equn. (IV.17) would take twice as much as comparison for computing \( \frac{\delta}{\delta X} D_s(X) \) and \( \frac{\delta}{\delta X} E_s(X) \). As a result computation of subgradients of our morphological operators \( \Upsilon(X) \) as described in equn. (III.3) takes twice as much as computation of derivatives of BTV regularization operator (II.9) and \( (2s + 1)^2 \) as much as computation of derivatives of TV regularization operator (II.8).

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